

VII. The Main Theorem

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VII. THE MAIN THEOREM

We are now able to characterize the definite spaces whose topology is admissible (Def. 1). Refer to Definition 21 for "type condition".

THEOREM 28 [20]. *Let \mathfrak{E} be a definite space in the sense of Definition 15. The following conditions are equivalent*

(i) $L_c(\mathfrak{E}) = L_s(\mathfrak{E})$ (cf. (1), (2), (3))

(ii) $L_c(\mathfrak{E}) = L_{\perp\perp}(\mathfrak{E})$ ("the topology is admissible", Def. 1)

(iii) k is complete and \mathfrak{E} is the completion of a \aleph_0 -dimensional space spanned by an orthogonal basis that satisfies the type condition.

Proof. (i) \Rightarrow (ii) holds trivially because $L_s \subseteq L_{\perp\perp} \subseteq L_c$ by continuity of the form; (ii) \Rightarrow (iii) was carried out in Chapter V. Just as in [18] we can establish (iii) \Rightarrow (i). Let $\mathfrak{U} \in L_c(\mathfrak{E})$. Pick a maximal orthogonal family $(v_i)_{i \in I}$ in \mathfrak{U} and extend it to a maximal orthogonal family $(v_i)_{i \in J}$ in \mathfrak{E} . For $x \in \mathfrak{E}$ we have by Lemma 27 $x = x' + x''$ where $x' = \sum_I \langle x, v_i \rangle \langle v_i \rangle^{-1} v_i$ and $x'' = \sum_J \langle x, v_i \rangle \langle v_i \rangle^{-1} v_i$. Now $x' \in \overline{\mathfrak{U}} = \mathfrak{U}$. All that remains to be shown is $x'' \in \mathfrak{U}^\perp$. Now \mathfrak{U}^\perp is closed so it suffices to show that $v_i \in \mathfrak{U}^\perp$ for all $i \in J$. To this end pick $u \in \mathfrak{U}$ and decompose $u = u' + u''$ (analogous to the decomposition of x): $u'' = u - u' \in \mathfrak{U} - \mathfrak{U} = \mathfrak{U}$. Now $\langle u'', v_i \rangle = 0$ for all $i \in I$ so $u'' = 0$ since $(v_i)_{i \in I}$ is a maximal orthogonal family. From

$$0 = u'' = \sum_J \langle u, v_i \rangle \langle v_i \rangle^{-1} v_i$$

we obtain $\langle u, v_i \rangle = 0$ ($i \in J$). As $u \in \mathfrak{U}$ was arbitrary this says that $v_i \in \mathfrak{U}^\perp$ ($i \in J$).
 Q.E.D.

Remark 29. Let the definite space \mathfrak{E} be the completion of $\mathfrak{F} = k(e_i)_{i \in \mathbb{N}}, (e_i)_{i \in \mathbb{N}}$ an orthogonal family (that does not necessarily satisfy the type condition). If k is complete then \mathfrak{E} is isometric to the k -space $\widehat{\mathfrak{F}}$ of all sequences $(\lambda_i)_{i \in \mathbb{N}} \in k^\mathbb{N}$ such that $\lim_{\mathbb{N}} (2\phi\lambda_i + \phi\langle e_i \rangle) = \infty$ and equipped with the form $\langle (\lambda_i), (\mu_i) \rangle = \sum_{\mathbb{N}} \lambda_i \mu_i \langle e_i \rangle$. Indeed, the set $\widehat{\mathfrak{F}}$ is a definite k -space and the map $\Psi: (\lambda_i) \rightarrow \sum \lambda_i e_i$ is a well defined isometry $\widehat{\mathfrak{F}} \rightarrow \Psi(\widehat{\mathfrak{F}}) \subset \mathfrak{E}$. By the "infinite Pythagoras" we have $\ker \Psi = 0$; on the other hand, Lemma 16 shows that Ψ is also surjective.

Thus all definite spaces that carry an admissible topology are (by Theorem 28) of the kind invented by Keller.

Remark 30. By Theorem 28 the isometry type of a definite space with admissible topology is characterized by the sequence $(\langle e_i \rangle)_{i \in \mathbf{N}}$ where $(e_i)_{i \in \mathbf{N}}$ is a maximal orthogonal family in \mathfrak{E} . Conversely, for each $(\alpha_i) \in k^{\mathbf{N}}$ there is a definite space \mathfrak{E} with $L_c(\mathfrak{E}) = L_s(\mathfrak{E})$ admitting a maximal orthogonal family $(e_i)_{i \in \mathbf{N}}$ with $\langle e_i \rangle = \alpha_i$ ($i \in \mathbf{N}$) provided that

- (A) $\xi_i := \varphi \alpha_i \in \Gamma$ satisfies the (type-) condition expressed in (8)
- (B) The form $\langle \cdot, \cdot \rangle$ defined on $\mathfrak{F} := k(e_i)_{i \in \mathbf{N}}$ by $\langle e_i, e_j \rangle = 0$ ($i \neq j$), $\langle e_i \rangle = \alpha_i$ ($i \in \mathbf{N}$) is definite.

These two conditions are implemented by many fields. In order to satisfy (A) one may, e.g. pick fields of generalized formal power series that are complete under a valuation φ with group Γ a prescribed Hahn product [30, p. 31] with sufficiently many factors not 2-divisible, e.g. $\Gamma = \mathbf{Z}^{(\mathbf{N})}$ ordered antilexicographically. Let k be any field with (A) and $t \in \Gamma/2\Gamma$; set $\mathfrak{F}_t = \{\text{span } e_i \mid \varphi \alpha_i + 2\Gamma = t\}$. By (A) $\dim \mathfrak{F}_t < \infty$; furthermore

$$\mathfrak{F} = \bigoplus^\perp \{\mathfrak{F}_t \mid t \in \Gamma/2\Gamma\}.$$

In order to check whether the form $\langle \cdot, \cdot \rangle$ satisfies the triangle inequality on \mathfrak{F} it suffices to verify said inequality on each \mathfrak{F}_t . A. Fässler has given a handy criterium for $\langle \cdot, \cdot \rangle$ to be definite if Hahnproducts Γ are used, as indicated, to construct k with (A), [6, Lemma 15, 16].

VIII. APPENDIX: EXTENDING THE MAIN THEOREM TO THE CLASS \mathcal{E} OF NORM-TOPOLOGICAL SPACES

The arguments applied to the spaces in the class \mathcal{D} can be extended to a larger class \mathcal{E} . First we have (cf. Definition 15):

Definition 31. An infinite dimensional anisotropic quadratic space $(\mathfrak{E}; \langle \cdot, \cdot \rangle)$ over a $*$ -valued field $(k, *, \varphi, \Gamma)$ is called norm-topological if the sets $\mathcal{U}_\gamma := \{x \in \mathfrak{E} \mid \varphi \langle x \rangle > \gamma\}$ form a 0-neighbourhood basis of a vector space topology on \mathfrak{E} . Let \mathcal{E} be the class of all norm-topological spaces.

Definite spaces are norm-topological, obviously.

A proper subgroup Δ of Γ is *convex* (or *isolated*) if “ $0 \leq x \leq y$ & $y \in \Delta$ ” implies “ $x \in \Delta$ ”. If the subgroup $\Delta \subset \Gamma$ is convex then the factor group Γ/Δ is ordered by setting $\gamma + \Delta \leq \delta + \Delta$ iff $\gamma < \delta$ or $\gamma - \delta \in \Delta$; furthermore, $\varphi_\Delta: k \rightarrow \Gamma/\Delta \cup \{\infty\}$ defined by $\varphi_\Delta(\alpha) = \varphi(\alpha) + \Delta$ is a valuation (a “coarser valuation”) which yields the same topology on k as φ .