4. The group of a category and ring

The proof of the superperfectness of $M(\Lambda, F)$ originally obtained for this paper so closely resembled in spirit those in [17], [2] showing the Steinberg groups to be superperfect that I felt that the two classes of groups ought to admit a mutual generalisation allowing a single proof of both facts. A suitable construction for this purpose appears below; it may well be of wider interest.

Let $\mathscr{C}$ denote a small category and A an associative ring with identity 1. Then $G = G_{\mathscr{C}, A}$ is to be the group with the following presentation. Its generating set is $\mathrm{Mor}(\mathscr{C}) \times A$, whose elements are typically written f^a. However, if f is an identity morphism $1_X (X \in \mathrm{Ob}(\mathscr{C}))$ then in G $1_X^a = 1$ for any $a \in G$. The remaining relations take the following form (for arbitrary non-identity morphisms f, g, h and ring elements a, b).

$$f^a f^b = f^{a+b} \tag{i},$$

and, if the composition $h \circ g$ is not defined,

$$[g^a, h^b] = \begin{cases} (g \circ h)^{ab} & \text{if defined} \quad \text{(ii)}, \\ 1 & \text{otherwise} \quad \text{(iii)}. \end{cases}$$

(From (i) and (ii) we may deduce that if $h \circ g$, but not $g \circ h$, is defined, then $[g^a, h^b] = (h \circ g)^{-ba}$. Finally, if $h \circ g$ and $g \circ h$ are both defined, then no immediate inference is available.)

Examples

1. For arbitrary A and integer $n \geqslant 3$ let $\mathscr{C}$ have object set $\{1, \ldots, n\}$ and, whenever $i \neq j$, morphisms $x_{ij}: j \mapsto i$ and $x_{ji}: i \mapsto j$. Then the group G is the Steinberg group $\mathrm{St}_n(A)$ or, when $n = \infty$, $\mathrm{St}(A)$; our notation for elements is just the usual one.

2. Given a linearly ordered set Λ, let $\mathrm{Ob}(\mathscr{C}) = \Lambda$ and let the non-identity morphisms be $e_{\lambda\mu}: \mu \mapsto \lambda$ corresponding to the relation $\lambda < \mu$. It is clear that relations (i)-(iii) above are just (i)-(iii) of Lemma 3.3 with the element $1 + ae_{\lambda\mu}$ now rewritten as $e_{\lambda\mu}^a$. So we obtain $M(\Lambda, F)$ when the ring A is a field F.

3. Let $\mathscr{C}$ be the category with one object and a morphism f having

exactly n distinct non-identity iterates. Then $G_{\mathscr{C},\mathbf{Z}}$ is the free group on the n (possibly infinite) generators.

4. Suppose now that $\mathscr{C}$ has two distinct objects X and Y, with n morphisms from X to Y. In this case $G_{\mathscr{C},\mathbf{Z}}$ is the free abelian group on n (again, possibly infinite) generators. Alternatively, let $\mathscr{C}$ have an initial object and n others, with only the morphisms required by this property. This provides an example where the category can be varied without changing the resulting group.

5. Of course, 3. and 4. above coincide when $n = 1$. Taking this case and an arbitrary ring A gives us for G the additive group A_+ of A.

An interesting question suggested by the above might be to classify all groups which can arise from our construction. Incidentally, although we do not exploit this fact, the construction induces a bifunctor from the product of the category of small categories (suitably restricted so that its morphisms are those functors for which every identity morphism has a unique preimage) and that of rings to the category of groups.

We now relate the (homology of) group-theoretical properties of perfectness and superperfectness of G to combinatorial properties of $\mathscr{C}$. To do this, consider the following possible conditions on non-identity morphisms of $\mathscr{C}$.

(α) Every f has its domain and codomain distinct.

(β) Every f is a non-trivial composite; that is, $f = g \circ h$ for some g and h.

(γ) Given f and k with $k \circ f$ undefined, then $f = g \circ h$ for some g and h such that $k \circ h$ and $g \circ k$ are undefined.

In the presence of (α), condition (γ) evidently implies (β) as the case $k = f$.

(δ) Any commuting square

$$\begin{array}{ccc} H & \xrightarrow{h} & Z \\ \downarrow{\scriptstyle t} & & \downarrow{\scriptstyle g} \\ W & \xrightarrow{r} & Y \end{array}$$

(with distinct vertices) admits a diagonal $s: Z \to W$ (or $W \to Z$) such that the two triangles formed commute.

Observe that Example 1. satisfies all these conditions, although (γ) forces n to be at least 5. Of course Example 2. satisfies (α) and (so long

as Λ has cardinality exceeding 3) (δ), while (β) and (γ) are each equivalent to Λ being a dense ordering.

LEMMA 4.1. *For arbitrary* A, *the group* $G_{\mathscr{C},\mathbf{A}}$ *is perfect if and only if* $\mathscr{C}$ *satisfies* (α) *and* (β).

Proof. The two conditions imply that

$$f^a = (g \circ h)^a = [g^1, h^a] .$$

Thus every element is a commutator. Conversely, given f which cannot be expressed as a composite $g \circ h$ where $h \circ g$ is undefined, then the homomorphism

$$G \to A_+ , \quad k^a \mapsto \begin{cases} 1 & k \neq f, \\ a & k = f, \end{cases}$$

has non-trivial abelian image, so that G cannot be perfect.

THEOREM 4.2. *For arbitrary* A, *if* $\mathscr{C}$ *satisfies conditions* (α), (γ) *and* (δ) *above, then* $G = G_{\mathscr{C},A}$ *is a superperfect group.*

Proof. By (4.1), G is perfect. To show that G is superperfect, we must establish that every central extension over G splits. Now notice that any splitting σ of a given central extension $K \rightarrowtail E \overset{\phi}{\twoheadrightarrow} G$ must have

$$\sigma(f^a) = [\bar{g}^1, \bar{h}^a]$$

whenever $f = g \circ h$ (since by (α) $h \circ g$ is undefined), where $\bar{h}^a$ denotes the K-coset $\phi^{-1}(h^a)$ in E. By virtue of the centrality of K, the right-hand expression determines a single element of E, which has yet to be shown independent of the particular factorisation of f. So a candidate for σ is defined by making a specific choice of factorisation for each morphism and having the above expression determine the value of each $\sigma(f^a)$. In the course of the proof it emerges that any choice would suffice, so that no real decision is actually required.

The proof uses the following facts [2 p. 68].

LEMMA 4.3. *In any group* E, *for* $u, v, w \in E$,

$$\begin{aligned} & [[u, v], w] \\ &= [u, v]\,[w, v]\,[v, wu] && && \text{(a)}, \\ &= [[u, v], [w, v]]\,[[w, v], u] && \text{if} \quad [u, w] \in Z(E) && \text{(b)}, \\ &= 1 && \text{if also} \quad [v, w] \in Z(E) && \text{(c)}. \end{aligned}$$

Here (a) is obtained by outright multiplication, and (b) by substitution for $[v, wu] = [v, uw]$ in (a). Applying condition (γ) we deduce straightaway from (c) that

$$[\bar{f}^{\,a}, \bar{k}^{\,b}] = 1 \text{ whenever } f \circ k \text{ and } k \circ f \text{ are not defined}. \qquad \text{(d)}$$

We are now in a position to prove that σ is a homomorphism by checking that it respects relations of types (i), (ii) and (iii).

(i). Let us first show that

$$\sigma(f^a)\sigma(f^b) = \sigma(f^{a+b}).$$

Suppose that $f = g \circ h$; by rearranging (a) we may rewrite the left-hand expression as

$$\begin{aligned} &[\bar{g}^{\,1}, \bar{h}^{\,a}\bar{h}^{\,b}]\,[\bar{h}^{\,a}, [\bar{g}^{\,1}, \bar{h}^{\,b}]^{-1}] \\ &= [\bar{g}^{\,1}, \bar{h}^{\,a+b}]\,[\bar{h}^{\,a}, \bar{f}^{\,-b}], \end{aligned}$$

which from (d) equals $[\bar{g}^{\,1}, \bar{h}^{\,a+b}]$ as required. Since

$$\sigma(1) = [\bar{g}^{\,1}, \bar{h}^{\,0}] = 1$$

because $\bar{h}^{\,0} = K$ is central, note that this forces $\sigma(f^{-a})$ to be inverse to $\sigma(f^a)$.

(ii). This is immediate from (d) and the fact that $\sigma(1) = 1$.

(iii). We require (δ), whose notation and assumptions we adopt. Then, from (b) and (d),

$$\begin{aligned} [\bar{g}^{\,a}, \bar{h}^{\,b}] &= [[\bar{r}^{\,1}, \bar{s}^{\,a}], \bar{h}^{\,b}] \\ &= [[\bar{r}^{\,1}, \bar{s}^{\,a}], [\bar{h}^{\,b}, \bar{s}^{\,a}]]\,[[\bar{h}^{\,b}, \bar{s}^{\,a}], \bar{r}^{\,1}] \\ &= [\bar{g}^{\,a}, (\overline{s\circ h})^{-ab}]\,[(\overline{s\circ h})^{-ab}, \bar{r}^{\,1}] \\ &= [\bar{r}^{\,1}, (\overline{s\circ h})^{-ab}]^{-1} \\ &= [\bar{r}^{\,1}, \bar{t}^{\,ab}], \end{aligned}$$

the last equality following from the arguments in (i). The case $a = 1$ shows σ to be independent of the choice of factorisation of $f = g \circ h$ and so well-defined. This is immediate for two factorisations through distinct objects as in (δ). Otherwise one has to use (γ) with $k = g$ in order to

construct such a pair of factorisations. This construction combines with σ being well-defined, to yield (iii).

Hence σ is a homomorphism after all, and the proof is complete.

Appendix — Homotopy groups of function spaces

The aim here is to determine the homotopy type of certain function spaces which are needed in § 1 above. (All function spaces are to have the compact-open topology.) The literature on this topic is bedevilled by the requirement of local compactness of X for exponential correspondence between maps $W \times X \to Y$ and $W \to Y^X$. As a result much of it seems to divide into two camps: those who state the facts we need in unnecessary speciality, and those who, presumably having missed the point entirely, claim undue generality.

Fortunately, it is possible to vary the local compactness assumption a little (just enough): in the present context of studying homotopy groups, the space W will be a sphere, thus compact. Now exponential correspondence still holds whenever $W \times X$ is a k-space (compactly generated space), and for compact W this occurs whenever X itself is a k-space [8].

Proposition A.1. *Suppose that, for some n, Y has $\pi_j(Y) = 0$ for $j > n$ and $\pi_1(Y)$ acting nilpotently on $\pi_j(Y)$ for $n - m < j \leqslant n$, with X an m-connected k-space. Then, for all $i \geqslant n - m$,*

$$\pi_i\big((Y, y_0)^{(X, x_0)}\big) = 0 .$$

Proof. By the preceding remarks, we consider maps

$$S^i \times (X, x_0) \to (Y, y_0) .$$

Let $g : (X, x_0) \to (Y, y_0)$ lie in the relevant path-component of the function space. Then all maps $S^i \times X \to Y$ under investigation have to restrict to $* \vee g : S^i \vee X \to Y$. Obstructions to deforming an arbitrary extension $f : S^i \times X \to Y$ of $(* \vee g)$ to $g \circ pr_2 : S^i \times X \to Y$ lie in the cohomology groups

$$H^q\big(S^i \times X, S^i \vee X ; \{\pi_q(Y)\}\big) \cong \tilde{H}^{q-i}\big(X ; \{\pi_q(Y)\}\big) .$$

Thus the only possible non-trivial obstructions lie in dimensions $q = i + 1, \ldots, n$. So when $m = 0$, $i \geqslant n$ suffices. More generally, let P_{n-m} be the first (lowest) space in the Postnikov tower for Y which is $(n-m)$-