§6. Hyperbolic Spaces

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vanishing of a polynomial. But a screw-motion σ has a fixed point if and only if the translation vector is perpendicular to the axis of the rotation. Since the axis of a rotation $(a_{ij}) \in SO_3$ is parallel to $(a_{32}-a_{23}, a_{13}-a_{31}, a_{21}-a_{12})$, σ has a fixed point if and only if $v_1(a_{32}-a_{23}) + v_2(a_{13}-a_{31}) + v_3(a_{21}-a_{12}) = 0$. This completes the proof of Theorem 1 (a) for $\mathbf{R}^n$.

§ 6. Hyperbolic Spaces

Here we meet a case where the existence of a free, fixed-point free group of isometries having rank 2 does not imply the existence of such a group having uncountable rank. The hyperbolic plane is such a space.

If H^2 is identified with the upper half-plane of $\mathbf{C}$, then $G(H^2)$ corresponds to linear fractional transformations $z \mapsto \dfrac{az+b}{cz+d}$, where a, b, c, d are real and $ad - bc \neq 0$. Since it may be assumed that $ad - bc = 1$, this group is isomorphic to $PSL_2(\mathbf{R})$. A nonidentity element of $PSL_2(\mathbf{R})$ is called elliptic, parabolic, or hyperbolic according as the absolute value of its trace is less than, equal to, or greater than two; the nonidentity elements of $G(H^2)$ with a fixed point in H^2 correspond to the elliptic elements of $PSL_2(\mathbf{R})$. See [18] for more details about this interpretation of $PSL_2(\mathbf{R})$. The following theorem clarifies the situation regarding fixed-point free subgroups of $G(H^2)$.

Theorem 3. (Siegel) *If F is a free subgroup of $PSL_2(\mathbf{R})$ then F is discrete if and only if F has no elliptic elements.*

Theorem 3 is a rephrasing of the result of [34] (see also [15]). An elementary proof appears in [41]. The forward direction is an immediate consequence of the fact that the nondiscrete cyclic subgroups of $PSL_2(\mathbf{R})$ are precisely the ones generated by an elliptic element of infinite order. This fact also yields the reverse direction in the case when F is cyclic. Siegel gave an algebraic proof of the reverse direction for noncyclic free groups. This can also be obtained by first using techniques of Lie algebras to show that a nondiscrete, nonsolvable subgroup of $PSL_2(\mathbf{R})$ is dense in $PSL_2(\mathbf{R})$, and observing that the elliptics form an open set; this approach is due, independently, to A. Borel and D. Sullivan.

The forward (easy) direction of Theorem 3 yields a proof of the positive part of Theorem 1 (b) for H^2 (and hence for H^n, $n \geqslant 2$), since it implies that a discrete free group of rank two has no elliptic elements. Therefore any rank-two free subgroup of $SL_2(\mathbf{Z})$ is fixed-point free when viewed as a group of isometries of H^2. The simplest example of such a free group is the subgroup generated by $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$ (see [6]). (Recall also the result, mentioned in § 2, that $PSL_2(\mathbf{Z}) \cong \mathbf{Z}_2 * \mathbf{Z}_3$.)

Moreover, the reverse direction of Theorem 3 yields the negative part of Theorem 1 (b). For an uncountable subgroup of $PSL_2(\mathbf{R})$ is necessarily nondiscrete, and so an uncountable free subgroup must contain an elliptic element.

Let us point out why the perfect set technique of the previous sections breaks down in H^2. A matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in $SL_2(\mathbf{R})$ is elliptic if and only if $(a+d)^2 < 4$, which is a polynomial inequality rather than an equality. Therefore the elliptics do not form a closed set, and hence they cannot be the zero set of an analytic function.

But the method of § 4 easily yields Theorem 1 (c) for H^2. Simply let $R_w = f_w^{-1}(\{I\})$; R_w is the zero set of an analytic function. Therefore the method of § 4 yields a free subgroup of $SL_2(\mathbf{R})$ (and hence of $PSL_2(\mathbf{R})$) with a perfect set of free generators. This proves Theorem 1 (c) for H^2, since the entire action of $PSL_2(\mathbf{R})$ on H^2 is locally commutative: if two elliptics share a fixed point, then they have the same set of fixed points in $\mathbf{C} \cup \{\infty\}$, so they commute.

A large, free locally commutative subgroup of $G(H^2)$ immediately yields such a subgroup of $G(H^n)$, $n \geqslant 3$, but a stronger result, namely Theorem 1 (a), is true in these higher dimensions. Consider first the case $n \geqslant 4$. By considering H^4 as the upper half-space in $\mathbf{R}^4$, it is easy to see that there is a monomorphism of $G(\mathbf{R}^3)$ into $G(H^4)$; any isometry of $\mathbf{R}^3$ is extended to H^4 by fixing the additional coordinate. Since a fixed-point free isometry remains so, Theorem 1 (a) for H^4 (and hence for H^n, $n \geqslant 4$) is a consequence of the corresponding result for $\mathbf{R}^3$. This method fails in H^3 however, since $G(\mathbf{R}^2)$ has no non-Abelian free subgroup.

To prove Theorem 1 (a) for H^3, we shall use the facts (see [3]) that $G(H^3)$ is isomorphic to $PSL_2(\mathbf{C})$ and that the elliptic transformations have real trace. (The elliptics, i.e., those nonidentity transformations in $PSL_2(\mathbf{C})$ fixing a point in H^3, are precisely the transformations whose trace is real and lies in the open interval $(-2, 2)$.) It will be more convenient to work in $SL_2(\mathbf{C})$ and there is no loss in so doing, since a free subgroup of $SL_2(\mathbf{C})$ induces one in $PSL_2(\mathbf{C})$. As before, consider a word w in m variables, and define R_w to be $\{(\sigma_1, \ldots, \sigma_m) \in SL_2(\mathbf{C})^m : w(\sigma_1, \ldots, \sigma_m) \text{ is elliptic}\}$. We wish to show that R_w is nowhere dense, and to this end we consider the superset R_w^* of R_w defined by:

$$R_w^* = \{(\sigma_1, \dots, \sigma_m) \in SL_2(\mathbf{C})^m : \operatorname{trace}(w(\sigma_1, \dots, \sigma_m)) \in \mathbf{R}\}.$$

LEMMA. *R_w^* is a nowhere dense subset of $SL_2(\mathbf{C})^m$.*

Proof. We shall view $SL_2(\mathbf{C})^m$ as a connected real analytic submanifold of $\mathbf{R}^{8m}$. If $a_1, \dots, a_{8m}$ are the reals defining $\sigma_1, \dots, \sigma_m$, then there are polynomials $p_1, \dots, p_8$ in the a_i such that

$$w(\sigma_1, \dots, \sigma_m) = \begin{pmatrix} p_1 + ip_2 & p_3 + ip_4 \\ p_5 + ip_6 & p_7 + ip_8 \end{pmatrix}.$$

Therefore $(\sigma_1, \dots, \sigma_m) \in R_w^*$ if and only if $p_2 + p_8 = 0$. Since R_w^* is closed, if R_w^* failed to be nowhere dense then it would contain a nonempty open set. As in § 4, this implies that $p_2 + p_8$ is identically zero on $SL_2(\mathbf{C})^m$ or, equivalently, that the trace of $w(\sigma_1, \dots, \sigma_m)$ is real for all $\sigma_1, \dots, \sigma_m \in SL_2(\mathbf{C})$. This leads to a contradiction as follows.

A result of Magnus [19] and Neumann [30] (for a proof, see [20, § III.2]) states that the matrices $\rho = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$ and $\tau = \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}$ are free generators of a subgroup of $SL_2(\mathbf{Z})$ that consists only of the identity and hyperbolic elements. It follows that the same is true of the group generated by $\sigma_1, \sigma_2, \dots$, where $\sigma_i = \rho^i \tau^i$. Now, for $z \in \mathbf{C} \setminus \{\frac{1}{2}(1 \pm \sqrt{5}), \frac{1}{4}(1 \pm \sqrt{5})\}$, define $\rho(z)$ and $\tau(z)$ in $SL_2(\mathbf{C})$ by

$$\rho(z) = \frac{1}{1 + z - z^2} \begin{pmatrix} 1 & z \\ z & 1+z \end{pmatrix} \quad \text{and} \quad \tau(z) = \frac{1}{1 + 2z - 4z^2} \begin{pmatrix} 1+2z & 2z \\ 2z & 1 \end{pmatrix}.$$

Let $\sigma_i(z) = \rho^i(z)\tau^i(z)$. Then choose a region Ω in $\mathbf{C}$ so that $0, 1 \in \Omega$ but Ω does not contain any of the 4 real singularities, and define a complex analytic function f on Ω by $f(z) = \operatorname{trace}(w(\sigma_1(z), \dots, \sigma_m(z)))$. The assumption on w of the previous paragraph, together with the Open Mapping Theorem applied to f, yields that f is constant on Ω. But $f(0) = 2$ and $f(1)$ is the trace of a nonidentity element of the Magnus-Neumann group, whence $f(1) \neq 2$, a contradiction. Alternatively (as pointed out by a referee), one can obtain a contradiction by using Theorem 1 and Remark 4 of [5] to obtain that $f_w(SL_2(\mathbf{C})^m)$ has nonempty interior, whence the image of trace $\cdot f_w$ has nonempty interior in $\mathbf{C}$. Therefore the image of trace f_w is not contained in $\mathbf{R}$.

This lemma implies that R_w is nowhere dense too, so we may apply Theorem 2 to the collection $\{R_w\}$, yielding Theorem 1 (a) for H^3. This completes the proof of Theorem 1.

§ 7. Geometrical Consequences

In this section we summarize some striking geometrical consequences of the existence of large free groups. The following theorem illustrates what can be done with locally commutative actions. Unlike the preceding sections, the results of this section all use the Axiom of Choice. We use $D \Delta E$ to denote $(D \backslash E) \cup (E \backslash D)$.

Theorem 4. *Suppose a free group, G, of rank $\kappa\,(\kappa \geqslant 2)$ is locally commutative in its action on X.*

(a) *If (and only if) $\kappa^\lambda = \kappa = |X|$, then there is a subset E of X such that for any $D \subseteq X$ with $|D| \leqslant \lambda$, there is some $\sigma \in G$ such that $\sigma(E) = E \Delta D$. In short, E is invariant under the addition and deletion of any λ points of X.*

(b) *X may be partitioned into κ sets, A_α, $\alpha < \kappa$, such that each A_α is G-equidecomposable with X using 2 pieces, i.e. for each α there are $\sigma_\alpha, \tau_\alpha \in G$ and $B_\alpha, C_\alpha \subseteq A_\alpha$ such that $\{B_\alpha, C_\alpha\}$ partitions A_α and $\{\sigma_\alpha(B_\alpha), \tau_\alpha(C_\alpha)\}$ partitions X. In short, X may be taken apart into pieces which may be rearranged to form κ copies of X.*

(c) *There is a subset E of X such that for any cardinal λ satisfying $3 \leqslant \lambda \leqslant \kappa$, X may be partitioned into λ G-congruent pieces, each of which is G-congruent to E. In short, E is, simultaneously, a third, a quarter, ..., a κ'th part of X.* (If the action is fixed-point free, then $\lambda = 2$ is also permitted — see Theorem 6.)

Parts (b) and (c) of this theorem are applications of a more general fact about locally commutative actions of a free group, which is described following Theorem 6.

Theorem 1 shows that all parts of the preceding theorem, with $\kappa = 2^{\aleph_0}$, apply to S^n, L^n and $H^n\,(n \geqslant 2)$ and $\mathbf{R}^n\,(n \geqslant 3)$, where G is either $G(X)$ or, in the case of L^n, the group of all isometries. Note that, since $(2^{\aleph_0})^{\aleph_0} = 2^{\aleph_0}$, part (a) yields a set that is invariant under the addition or deletion of countably many points. Because the existence of large free locally commutative groups was already known in most of these cases, so were the consequences by Theorem 4; only the cases of S^4 and L^4 are new.