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PARTITIONS INTO SQUARES

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ABSTRACT [1]). In following D. H. Lehmer (*Amer. Math. Monthly* 55 (1948), 476-481), the notation $P_k(n)$ stands for the number of partitions of the integer n into sums of k squares of positive integers in, say, non-decreasing order. D. H. Lehmer has determined the sets $S_{k,1}$ of integers n with $P_k(n) = 1$. Here the sets $S_{k,m}$ of integers satisfying $P_k(n) = m$ are determined. Next, the sets $S_{k,m}(x) \subset S_{k,m}$, with $n \leqslant x$ are described and exact or asymptotic formulae are obtained for their size.

1. INTRODUCTION. In 1948, D. H. Lehmer [8] studied the solutions of the equation $P_k(n) = 1$, where $P_k(n)$ is the number of *partitions* of the integer n into a sum of k squares of non-negative, non-decreasing integers. The number $P_k(n)$ of partitions differs from the number $r_k(n)$ of *representations* of n by k squares in that, for $r_k(n)$, we count representations as distinct if they differ by the order of the summands and the summands are squares of arbitrary (also negative) integers. If all summands of a partition counted by $P_k(n)$ are distinct and non-zero, then to it correspond $2^k k!$ representations counted by $r_k(n)$. If any one of the summands of a partition vanishes, or if not all are distinct, then to that partition correspond fewer than $2^k k!$ representations, but in any case, $P_k(n) \geqslant r_k(n)/2^k k!$.

In [8] Lehmer essentially solved the problem quoted above, except for $k = 3$. A contribution to the latter case was recently made by Bateman and the author (see [2]), but the problem is still not completely solved.

The purpose of the present paper is to discuss the more general question of the sets

$$S_{k,m} = \{n \mid P_k(n) = m\} \text{ and } S_{k,m}(x) = \{n \mid n \in S_{k,m}, n \leqslant x\},$$

as well as $| S_{k,m}(x) |$, the number of elements of $S_{k,m}(x)$.

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