# 1. The effect on mathematics

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things to be done more quickly, and in different ways, and the influence of fundamental concepts of informatics, in the forefront of which is found algorithmics.

## 1. The effect on mathematics

Mathematical concepts have always depended on methods of calculation and methods of writing. Decimal numeration, the writing of symbols, the construction of tables of numerical values all preceded modern ideas of real number and of function. Mathematicians calculated integrals, and made use of the integration sign, long before the emergence of Riemann's or Lebesgue's concepts of the integral. In a similar manner, one can expect the new methods of calculation and of writing which computers and informatics offer to permit the emergence of new mathematical concepts. But, already today, they are pointing to the value of ideas and methods, old or new, which do not command a place in contemporary "traditional" mathematics. And they permit and invite us to take a new look at the most traditional ideas.

Let us consider different ideas of a real number. There is a point on the line $R$, and this representation can be effective for promoting the understanding of addition and multiplication. There is also an accumulation point of fractions, for example, continued fractions giving the best approximation of a real by rationals. There is also a non-terminating decimal expansion. There is also a number written in floating-point notation. Experience with even a simple pocket calculator can help validate the last three aspects. The algorithm of continued fractions—which is only that of Euclid—is again becoming a standard tool in many parts of mathematics. Complicated operations (exponentiation, summation of series, iterations) will, with the computer's aid, become easy. Yet even these simplified operations will give rise to new mathematical problems: for example, summing terms in two different orders (starting with the largest or starting from the smallest) will not always produce the same numerical result.

Again, consider the notion of function. Teaching distinguishes between, on the one hand, elementary and special functions—that is, those functions tabulated from the 17th to the 19th century—and, on the other, the general concept of function introduced by Dirichlet in 1830. Even today, to "solve" a differential equation is taken to mean reducing the solution to integrals, and if possible to elementary functions. However, what is involved in functional equations is the effective calculation and the qualitative study of solutions. The functions in which one is interested therefore are calculable functions and no longer only those which are tabulated. The theories of approximation and of the superposition of functions—developed well before computers—are now validated. The field of elementary functions is extended, and functions of a non-elementary nature are introduced naturally through the discretisation of non-linear problems. Informatics, too, compels us to take a new look at the notion of a variable, and at the link between symbol and value. This link is strongly exploited in mathematics (for example, in the symbolism of the calculus). In informatics, the necessity of working out, of realising the values has presented this problem in a new way. The symbolism of functions is not entirely transferable. This has resulted in computer languages of different types: thus the notion of a variable in LISP does not correspond exactly to that in some other languages in which variables have values.

For the last of our examples let us consider sets of points linked to dynamical systems, iteration of transformations or stochastic processes. The use of computers has brought new life to their study, both by physicists and mathematicians, and has given rise to a new terminology: for example, strange attractors, fractals.

From these examples it can be seen that computers and informatics have *stimulated* new research, *restored* to the mathematician's consideration questions recently neglected but previously studied over a long period of time, and *made possible* the study of new questions. We hope that as a result of the discussions connected with the ICMI study new light will be cast on each of these aspects.

There has always been an experimental side to mathematics. Euler insisted on the rôle of observation in pure mathematics: "the properties of numbers that we know have usually been discovered by observation, and discovered well before their validity has been confirmed by demonstration ... It is by observation that we increasingly discover new properties, which we next do our utmost to prove". Computers have suddenly greatly increased our possibilities for observation and experimentation in mathematics. The solution of the non-linear wave equation, the soliton, was discovered by numerical experimentation before it became a mathematical object, and gave rise to a rigorous theory. In the iteration of rational transformations it is the plots obtained by computers which have guided recent research. An entirely new art of experimentation is developing in all branches of mathematics. Calculations which were formerly impracticable are now easily accomplished; it is now a question of working out an appropriate plan of action. Visualisations are possible and they form a unifying bond between mathematicians in offering them subjects for study on which specialists from different disciplines can unite to work. There has been a considerable increase in the number and variety of stimuli which allow, indeed encourage, one to query and investigate their mathematical nature in order to establish and appreciate their inter-relationships. An awareness of these new possibilities has for some years penetrated research mathematics. Only on rare occasions has it been allowed to influence and infiltrate our teaching. However, these possibilities for experimentation, now practicable on a large scale, are most full of promise for the renewal and improvement of the teaching of mathematics.

Mathematics is also, and will remain, a science of proof. But the status of proof is not immutable. The level of rigour and degree of formalisation depends upon time and place. For some twenty years the fashion was for non-constructive proofs of existence theorems: methods of ideals for g.c.d., pigeon-hole principle for rational approximations, axiom of choice for functional analysis, probabilistic methods without explicit constructions, etc. Today the point of view has changed. Whenever possible, one makes use in a proof of an algorithm which permits one to obtain effectively the object sought.

Computers have had another effect on the status of proof, as has been shown in the celebrated case of the four-colour theorem. Until now, the most long and involved proofs were edited and published and the reader had access to and control over any exterior information required (tables, references). In principle, a mathematician working alone was supposed to be able to follow and verify every step of a proof thanks to this method of presentation. Now new types of proof have appeared: numerical proofs in which there occur numbers of a size and in quantities which preclude their being manipulated by hand, and algorithmic proofs dependent on the effectiveness and correctness of the algorithms. Computer-aided proofs have produced the need, therefore, for a new form of professional practice. This does not yet seem to have been codified. Doubtless it will be in the future.

Algorithms have played an important rôle in mathematics since Euclid, and even more since the birth of algebra. They constitute the most important mathematical constituent of informatics. We have referred to the rôle of algorithms as tools in proof and we are all aware that they are essential tools in calculation. Now, however, they are becoming more and more a study in themselves. Fascinating questions now arise on space and time complexity—on how to formulate algorithms so as to minimize computer storage space and running time—and on the development of algorithms suitable for processors running in parallel. To take but one example, by mathematical ingenuity the time complexity for the fast Fourier transform algorithm has been reduced from $n^2$ to $n \log n$, which is of considerable practical importance for large values of $n$. There are other problems concerned with the effectiveness of algorithms, their correctness and the way in which they can be elaborated. We note, as an example, the rôle of invariants and fixed points when establishing the correctness of algorithms.

One must also stress how algorithms are increasingly being called upon to play a central rôle in society: they arise in business and commerce, in technology and in automation. Mathematical problems arise then in many new domains, and mathematical methods have an increasingly far-reaching applicability.

Finally, from now on symbolic systems will enable the computer user to carry out difficult calculations within algebra and analysis. The possibilities raised are enormous, and one must take the measure of the actual performance of such systems and of their rôle in research mathematics, as well as of the influence they should have on the teaching of mathematics at the university and pre-university levels. Informatics, for example, extends the field of mathematical research on formal calculus.

## 2. The effect of computers on curricula

Curricula are generally the product of a long tradition, and their evolution is governed by two principal factors: the needs of society and the state of the discipline. The needs of society are very diverse: in each country, studies prepare for different professions, each of which has its own demands; between different countries there will be varying priorities. *A priori*, social needs introduce into curricula an element of diversity and even of divergence. On the other hand, reference to the discipline of mathematics itself is usually a unifying factor, when the specialists agree amongst themselves on what is essential content. And this unity also responds to a social need, to have a common body of knowledge and a shared language.

We have therefore to consider two major series of questions: the first relating to the expressed needs of society, to local experiences, to national policies; the second relating to new possibilities, to the adaptations which will have to be made as a result of new requirements, to choices prompted by the present state of knowledge and technique.