4. Distinguished Bases for the Bimodular Singularities

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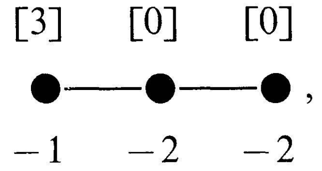
isomorphic Milnor lattices and by [8] isomorphic monodromy groups. These are the singularities given by

$$z^3 + x^4 + y^{36}$$

and

$$z^2 + y(x^{12} + y^{18}).$$

The resolution graph is in both cases



where the number in brackets denotes the genus, the other the selfintersection number of the corresponding cycle. Here $(\mu_0, \mu_+, \mu_-) = (6, 42, 162)$. However, the orders of the classical monodromy operators are 36 resp. 38.

4. Distinguished Bases for the Bimodular Singularities

We have seen in the last section that there are bimodular singularities which have the same Dynkin diagrams with respect to weakly distinguished bases, but not with respect to distinguished bases. We now turn our attention to the sets $\mathscr{B}^*$ for these singularities. Let us first look at the unimodular case. All exceptional unimodular singularities have a weakly distinguished basis with a Dynkin

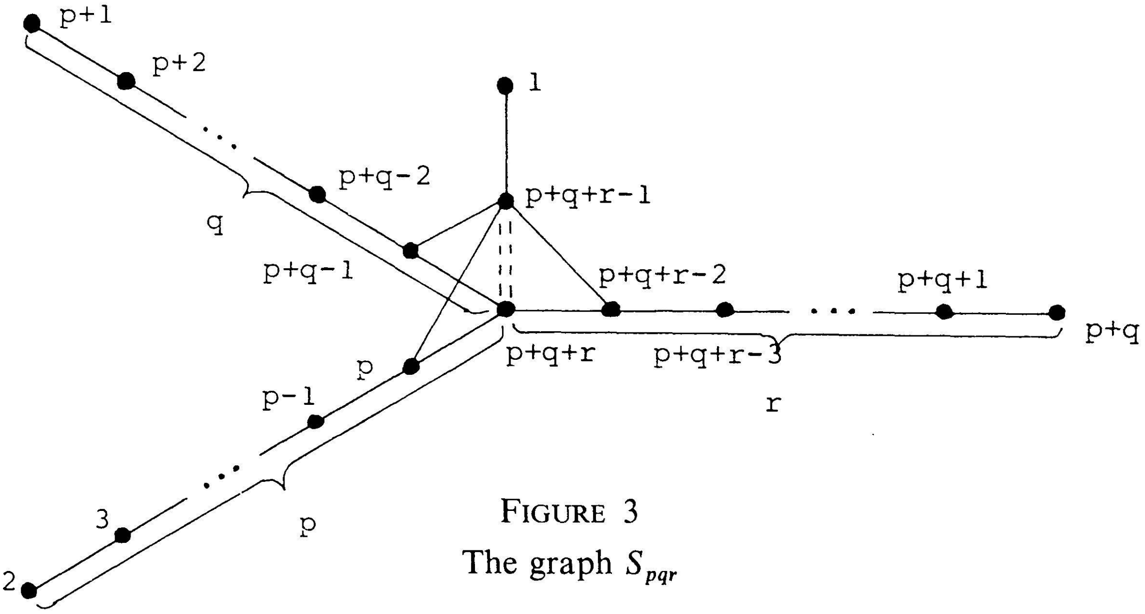


FIGURE 3
The graph S_{pqr}

diagram in Gabrielov's canonical form given by the graph of Fig. 2 setting $d = e = 1$. One can show that these graphs provided with the numbering of Fig. 3 also correspond to distinguished bases. (The graph with this numbering is obtained from the graph in [7, Abb. 15] by the following transformations: We indicate only the transformations for the first branch, the other branches are treated in an analogous manner: $\beta_7, \beta_6, \beta_5, \beta_4, \beta_3; \beta_8, \beta_7, \beta_6, \beta_5, \beta_4; \dots; \beta_{p+4}, \beta_{p+3}, \beta_{p+2}, \beta_{p+1}, \beta_p; \gamma_2, \gamma_3, \dots, \gamma_{p-1}$). We call this graph S_{pqr}.

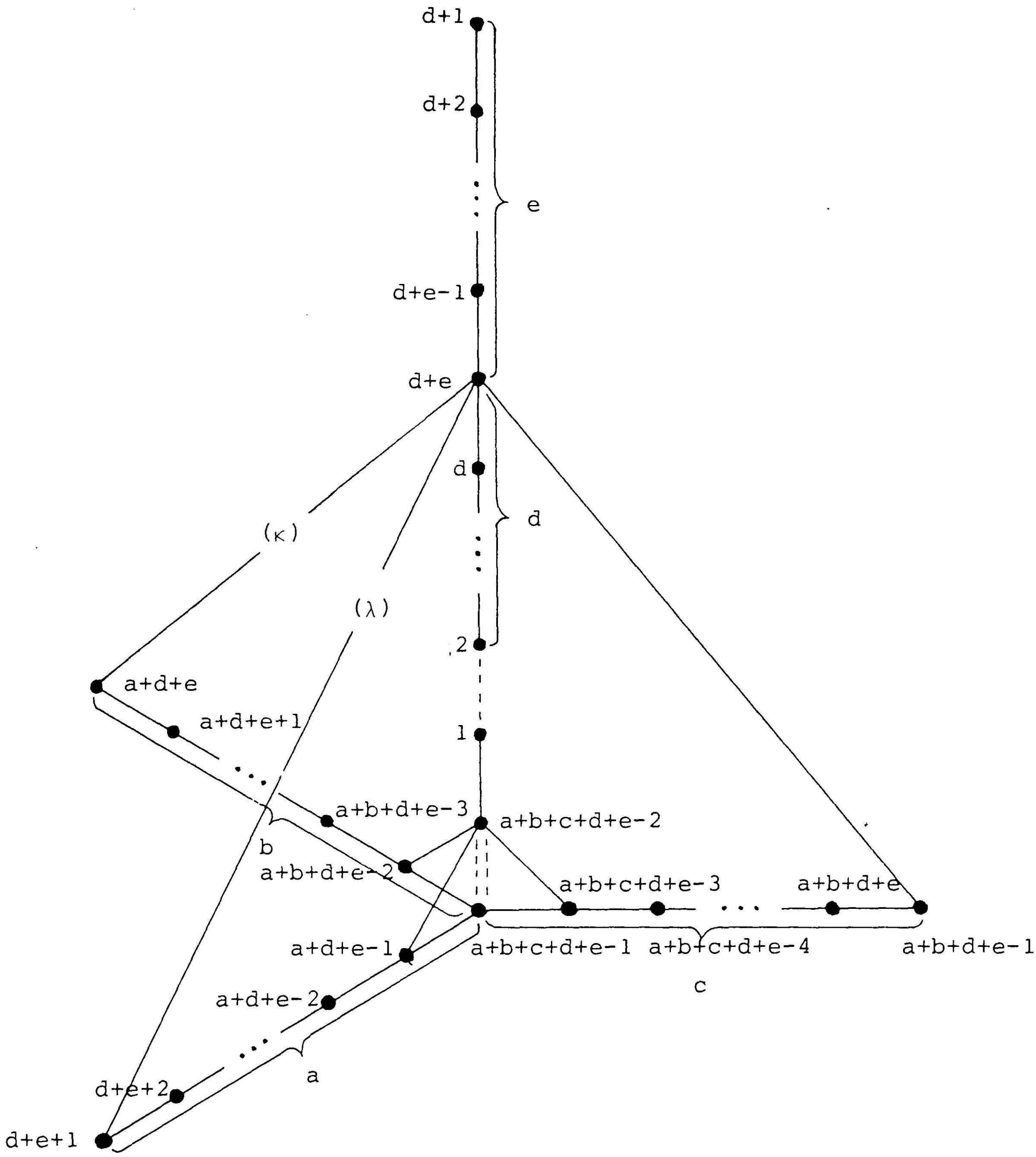


FIGURE 4
The graph $R^{\kappa\lambda}_{abcde}$

A natural form for the Dynkin diagrams of elements of $\mathscr{B}^0$ for the bimodular singularities E_{18} and Q_{18} is given in Fig. 2. Not all bimodular singularities have a Dynkin diagram of this type, one has to allow additional edges between e_4 and e_5 and between e_6 and e_7 (see [7]). But one can show by the methods introduced later in this section that none of the diagrams of Fig. 2/Table 1 equipped with any numbering corresponds to a distinguished basis of any of these singularities. However, there are elements of $\mathscr{B}^*$ with a Dynkin diagram of a form which is very close to the form of Fig. 2: one has to add only one dotted edge to this diagram. More precisely we have the following theorem:

THEOREM 4.1. *All bimodular singularities have a distinguished basis with the Dynkin diagram* $R^{\kappa\lambda}_{abcde}$ *shown in Fig. 4, where the values* $\kappa, \lambda, a, b, c, d, e$ *are given in Table 2.*

The graph $R^{\kappa\lambda}_{abcde}$ is defined for $a, b, c \geqslant 2$, $d, e \geqslant 1$, $\kappa, \lambda \in \{0, 1\}$ and $\lambda \leqslant \kappa$. Here $\kappa = 0(1)$ means that there is no edge (is an edge) between e_{d+e} and e_{a+d+e} ($\lambda = 0(1)$ analogously). In Table 2 the values of d and e can be interchanged and for $\kappa = d = e = 1, \lambda = 0$ all values $b', c' \geqslant 2$ with $b' + c' = b + c$ (b, c in the table) are possible. Finally $i, j, k \geqslant 0$.

We shall examine the graph $R^{\kappa\lambda}_{abcde}$ more closely. Such a labelled weighted graph defines in an obvious way a lattice and a basis in this lattice (setting $\langle e_i, e_i\rangle = -2$ for all vertices e_i). The rank $rk(R^{\kappa\lambda}_{abcde})$ and discriminant $\operatorname{disc}(R^{\kappa\lambda}_{abcde})$ of the lattice defined by $R^{\kappa\lambda}_{abcde}$ are given by the following general formulas:

$$rk(R^{\kappa\lambda}_{abcde}) = a+b+c+d+e-1 = \mu,$$

$$\operatorname{disc}(R^{\kappa\lambda}_{abcde}) = (-1)^{\mu-1}.$$

$$\{[(1+\kappa+\lambda)c-1]\,(ab-a-b) - (1+\kappa+\lambda)ab - \kappa a(c+1)$$

$$-\lambda b(c+1) + (\kappa-\lambda)c\}\ de - [(c-1)ab-c(a+b)]\,(d+e).$$

Such a graph R also defines a Coxeter element C_R which is by definition the product of reflections corresponding to the vertices e_i,

$$C_R = s_{e_1} \circ \ldots \circ s_{e_\mu}.$$

In the case that the graph is the Dynkin diagram of a distinguished basis, the Coxeter element C_R corresponds to the classical monodromy operator. Now by [2, Ch. V.6, Exercice 3] the characteristic polynomial $P_R(t)$ of C_R can be computed as follows

TABLE 2

Sing.	κ λ	a	b	c	d	e
$J_{3,i}$	0 0	2	3	8+i	2	2
	0 0	2	3	8	2+i	2
$Z_{1,i}$	0 0	2	4	6+i	2	2
	0 0	2	4	6	2+i	2
$Q_{2,i}$	0 0	3	3	5+i	2	2
	0 0	3	3	5	2+i	2
$W_{1,i}$	0 0	2	5	5+i	2	2
	1 0	2	6	6	1+i	1
$W^{\#}_{1,i}$, i>0	0 0	2	5	5	2+i	2
i=j+k-8	1 0	2	2+j	2+k	1	1
	1 0	2	5	7	1+i	1
$S_{1,i}$	0 0	3	4	4+i	2	2
	1 0	3	5	5	1+i	1
$S^{\#}_{1,i}$, i>0	0 0	3	4	4	2+i	2
i=j+k-6	1 0	3	2+j	2+k	1	1
	1 0	3	4	6	1+i	1
$U_{1,i}$	1 0	4	2+j	2+k	1	1
i=j+k-5	1 0	4	4	5	1+i	1
$U_{1,1}$	1 1	4	5	5	1	1

Sing.	κ λ	a	b	c	d	e
E_{18}	0 0	2	3	9	2	3
	0 0	2	3	8	3	3
E_{19}	0 0	2	3	10	2	3
	0 0	2	3	9	2	4
	0 0	2	3	8	3	4
E_{20}	0 0	2	3	11	2	3
	0 0	2	3	9	2	5
	0 0	2	3	8	3	5
Z_{17}	0 0	2	4	7	2	3
	0 0	2	4	6	3	3
Z_{18}	0 0	2	4	8	2	3
	0 0	2	4	7	2	4
	0 0	2	4	6	3	4
Z_{19}	0 0	2	4	9	2	3
	0 0	2	4	7	2	5
	0 0	2	4	6	3	5
Q_{16}	0 0	3	3	6	2	3
	0 0	3	3	5	3	3
Q_{17}	0 0	3	3	7	2	3
	0 0	3	3	6	2	4
	0 0	3	3	5	3	4
Q_{18}	0 0	3	3	8	2	3
	0 0	3	3	6	2	5
	0 0	3	3	5	3	5
W_{17}	0 0	2	5	6	2	3
	1 0	2	6	7	1	2
W_{18}	0 0	2	5	7	2	3
	1 0	2	7	7	1	2
	1 0	2	6	7	1	3
S_{16}	0 0	3	4	5	2	3
	1 0	3	5	6	1	2
S_{17}	0 0	3	4	6	2	3
	1 0	3	6	6	1	2
	1 0	3	5	6	1	3
U_{16}	1 0	4	5	5	1	2
	1 1	5	5	5	1	1

$$P_R(t) = \det(t\,.\,1 - C_R) = \begin{vmatrix} 1+t & -\langle e_1, e_2\rangle\, t & \cdots & -\langle e_1, e_\mu\rangle\, t \\ -\langle e_2, e_1\rangle & 1+t & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & & \cdot & \cdot \\ \cdot & & \cdot & \cdot \\ -\langle e_\mu, e_1\rangle & \cdots & & 1+t \end{vmatrix}$$

In particular

$$P_R(1) = (-1)^\mu \operatorname{disc}(R)\,.$$

One can associate a directed graph R' to R as follows: Replace each edge between vertices e_i and e_j with $i < j$ by an arrow of the same type (dotted or not) pointing to e_j, and omit the numbering of the vertices. Then $P_R(t)$ depends only on R' and not on the special admissible numbering. Using the methods of [6], we have calculated $P_R(t)$ for $R = R^{\kappa\lambda}_{abcde}$ and obtained the following result. Let $I = \{a, b, c, d, e\}$ and for $J \subset I$ define ΣJ to be the formal expression

$$\sum_{j \in J} j\,.$$

Then the formal expression for $P_R(t)$ is

$$P_R(t) = (t-1)^{-5}\left(\sum_{\substack{J \subset I \\ \# J \leqslant 2}} \left(P_J(t)t^{\Sigma J} - P_J\left(\frac{1}{t}\right) t^{\mu+5-\Sigma J}\right)\right),$$

where

$$\begin{aligned}
P_\varnothing &= (1+\kappa+\lambda)t^4 + 3t^3 - 6t^2 + 4t - 1\,,\\
P_{\{a\}} &= -(1+\kappa)t^4 - (1-\kappa+2\lambda)t^3 + (3-\kappa+2\lambda)t^2 - 3t + 1-\lambda\,,\\
P_{\{b\}} &= -(1+\lambda)t^4 - (1+\kappa)t^3 + (3+\lambda)t^2 - (3-\kappa+\lambda)t + 1-\kappa\,,\\
P_{\{c\}} &= -(\kappa+\lambda)t^4 - 2t^3 + (2+\kappa+\lambda)t^2 - (1+\kappa+\lambda)t\,,\\
P_{\{d\}} &= P_{\{e\}} = -(1+\kappa+\lambda)t^4\,,\\
P_{\{a,b\}} &= t^4 - (1-\kappa-\lambda)t^3 - (\kappa+\lambda)t^2 + 2t - (1-\kappa-\lambda)\,,\\
P_{\{a,c\}} &= \kappa t^4 + (1-\kappa+\lambda)t^3 - (1+\lambda)t^2 + (1+\kappa)t + \lambda\,,\\
P_{\{b,c\}} &= \lambda t^4 + t^3 - (1-\kappa+2\lambda)t^2 + (1-\kappa+2\lambda)t + \kappa\,,
\end{aligned}$$

$$P_{\{a,\,d\}} = P_{\{a,\,e\}} = (1+\kappa)t^4 - (1+\kappa-2\lambda)t^3 + (1+k-2\lambda)t^2 + \lambda\,,$$
$$P_{\{b,\,d\}} = P_{\{b,\,e\}} = (1+\lambda)t^4 - (1-\kappa)t^3 + (1-\lambda)t^2 - (\kappa-\lambda)t + \kappa\,,$$
$$P_{\{c,\,d\}} = P_{\{c,\,e\}} = (\kappa+\lambda)t^4 + (2-\kappa-\lambda)t^2 - (2-\kappa-\lambda)t + 1\,,$$
$$P_{\{d,\,e\}} = (\kappa+\lambda)t^4 + t^3\,.$$

Now given the characteristic polynomial of the classical monodromy operator of a bimodular singularity, one can compute the values of $\kappa, \lambda, a, b, c, d, e$ for which the polynomial above coincides with it. In this way one gets

SUPPLEMENT TO THEOREM 4.1. *Table 2 (the remarks after Theorem 4.1 taken into account) contains for each bimodular singularity all possible values* $\kappa, \lambda, a, b, c, d, e$ *such that the graph* $R^{\kappa\lambda}_{abcde}$ *is a Dynkin diagram with respect to a distinguished basis of the singularity.*

The graph S_{pqr} is related to the graph $R^{\kappa\lambda}_{abcde}$ in the following way. The group

$$Z^* = Z_\mu \rtimes (\mathbf{Z}/2\mathbf{Z})^\mu$$

acts also on the set of all labelled graphs weighted by ± 1 with μ vertices. We denote equivalence under Z^* by $\sim$. Then

$$R^{00}_{abc1e} \sim R^{00}_{a,\,b,\,c+1,\,1,\,e-1} \qquad (e \geqslant 2)$$
$$R^{00}_{abc11} \sim S_{a,\,b,\,c+1}$$
$$R^{00}_{abcd1} \sim R^{00}_{a,\,b,\,c+1,\,d-1,\,1} \qquad (d \geqslant 2)$$

(*Proof:* $\beta_3, \beta_4, \ldots, \beta_\mu, \beta_\mu, \beta_{\mu-1}, \gamma_{\mu-2}$).

Therefore Theorem 4.1 and the supplement above imply in particular that none of the bimodular singularities has a distinguished basis with a Dynkin diagram of type S_{pqr}.

A closer study of Table 2 yields the following observation, with which we want to conclude. Let $R^{\kappa\lambda}_{abcde}$ be a graph of a singularity X of Table 2. Substract 1 from one of the following parameters:

c, d, e	for the E/J-, Z-, Q- series
b, c, d, e	for the W-, S- series
a, b, c, d, e	for the U- series

such that the new parameters $\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}, \tilde{e}$ still satisfy $\tilde{a}, \tilde{b}, \tilde{c} \geqslant 2$, $\tilde{d}, \tilde{e} \geqslant 1$. Then either $R^{\kappa\lambda}_{\tilde{a}\tilde{b}\tilde{c}\tilde{d}\tilde{e}}$ is again a graph of Table 2, say of the singularity Y, and we relate X and Y by an arrow $X \to Y$. Or it is equivalent under Z^* to a graph of the form

S_{pqr} which does not correspond to a distinguished basis of any unimodular singularity. So the graphs of the bimodular singularities cannot be simplified by the action of Z^* to a graph S_{pqr}, but the graphs immediately "below" them can. On the other hand the relations one gets by the arrows are exactly the adjacency relations of Laufer [15] between bimodular singularities with the difference of the Milnor numbers being equal to 1.