§5. Algebraic Structures on P.L. Manifolds

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To prove that P.L. manifolds are homeomorphic to algebraic sets we first define a class of stratified spaces (A-spaces) which admit "topological resolutions" to smooth manifolds, then we prove that these spaces are homeomorphic to algebraic sets. Then the result is achieved by showing that this class is big enough to contain all P.L. manifolds.

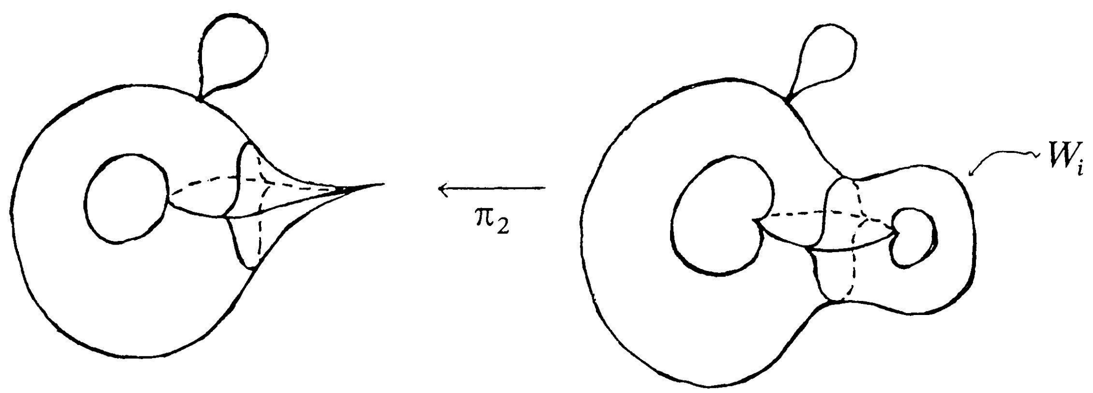
Define A_0-spaces to be smooth manifolds. Inductively let *A_k-spaces* to be spaces in the form $M = M_0 \bigcup_{\partial} N_i \times \text{cone}(\Sigma_i)$ where M_0 is an A_{k-1}-space and Σ_i are boundaries of compact A_{k-1}-spaces and N_i are smooth manifolds. The union is taken along codimension zero subsets of ∂M_0 and $N_i \times \Sigma_i \subset N \times \text{cone}(\Sigma_i)$. We define

$$\partial M = (\partial M_0 - \bigcup N_i \times \Sigma_i) \cup \bigcup \partial N_i \times \text{cone}(\Sigma_i),$$

hence boundaries of A_k-spaces are A_k-spaces. We call a space an *A-space* if it is an A_k-space for some k. If in the above definition we also assume that each Σ_i is a P.L. sphere then we call the resulting A-space *A-manifold.* A-manifolds are P.L. manifolds equipped with above special structure. A-spaces are more general than A-manifolds, for example they don't have to be manifolds.

A-spaces are constructed so that they can be "topologically" resolved. If M is an A_k-space $M_0 \cup \bigcup N_i \times \text{cone}(\Sigma_i)$, we can choose compact A_{k-1} spaces W_i with $\partial W_i = \Sigma_i$. We can construct the obvious A_{k-1} space $\tilde{M}_{k-1} = M_0 \cup \bigcup N_i \times W_i$. There is the obvious map $\pi_k : \tilde{M}_{k-1} \to M$ which is identity on M_0 and takes $N_i \times W_i$ to $N_i \times \text{cone}(\Sigma_i)$ by collapsing $N_i \times$ *spine*(W_i) onto $N_i \times$ point. By iterating this process we get a resolution tower:

$$\tilde{M} = \tilde{M}_0 \xrightarrow{\pi_1} \tilde{M}_1 \xrightarrow{\pi_2} \dots \to \tilde{M}_{k-1} \xrightarrow{\pi_k} M$$



an A_2 space M $\qquad\qquad$ $\tilde{M}_1$

with $\tilde{M}$ a smooth manifold. In fact by proving a generalized version of Proposition 4.2 we can adjust W_i so that each W_i has a spine S_i consisting of transversally intersecting A_{k-1} spaces without boundaries, and then each map π_k collapses $N_i \times S_i$ to $N_i \times$ point. This makes $\pi : \tilde{M} \to M$, where $\pi = \pi_k \circ \ldots \circ \pi_1$, very much analogous to a multiblowup.

THEOREM 5.1 ([AK_6]). *The interior of any compact A-space is homeomorphic to a real algebraic set. Furthermore the natural stratification on this algebraic set coincides with the stratification of the A-structure.*

Theorem 5.3 tells that the class of A-spaces contain all compact P.L. manifolds hence:

COROLLARY 5.2. *The interior of any compact P.L. manifold is P.L. homeomorphic to a real algebraic set.*

The idea of the proof Theorem 5.1 goes as follows. First define $\mathcal{O}_*(V)$, a bordism group for an algebraic set V. It is the usual bordism group of maps of A-spaces into V modulo the subgroup generated by maps $X \times N \to N \to V$ where X is an A-space, N is a nonsingular algebraic set and the map is the projection followed by an entire rational map $N \to V$. Then inductively we prove a generalized version of Theorem 2.8: that is if $M \subset V$ is an imbedding of a compact A-space without boundary into a nonsingular algebraic set V such that M represents 0 in $\mathcal{O}_*(V)$, then M can be moved to an algebraic subset Z of $V \times \mathbf{R}^n$ by a small isotopy (for some n). This implies the proof of Theorem 5.1 (by taking $V = \mathbf{R}^n$). Because one point compactification of an interior of a compact A-space is a compact A-space without boundary hence is homeomorphic to an algebraic set by above (and use Proposition 3.1 (b)).

Roughly the proof of the above claim proceeds as follows. Let $M = M_0 \cup N \times \text{cone}(\Sigma) \subset V$ then the bordism condition on M implies that $[N] \in \eta_*^A(V)$, so by Theorem 2.8 we can assume that N is a nonsingular algebraic subset of $V \times \mathbf{R}^m$ for some m. Define $B_1(V \times \mathbf{R}^m, N) = B(V \times \mathbf{R}^m \times \mathbf{R}, N \times 0)$, then this contains a natural nonsingular algebraic subset $N_1(V \times \mathbf{R}^m, N) = B(N \times \mathbf{R}, N \times 0)$ which is diffeomorphic to N. By continuing in this fashion let

$$B_k(V \times \mathbf{R}^m, N) = B\big(B_{k-1}(V \times \mathbf{R}^m, N) \times \mathbf{R}, N_{k-1}(V \times \mathbf{R}^m, N) \times 0\big),$$
$$N_k(V \times \mathbf{R}^m, N) = B\big(N_{k-1}(V \times \mathbf{R}^m, N) \times \mathbf{R}, N_{k-1}(V \times \mathbf{R}^m, N) \times 0\big).$$

Then we get a generalized algebraic multiblowup $\pi_k : B_k(V \times \mathbf{R}^m, N) \to V \times \mathbf{R}^m$

such that $\pi_k^{-1}(N)$ is a union of codimension one submanifolds $\bigcup S_i$ in general position and

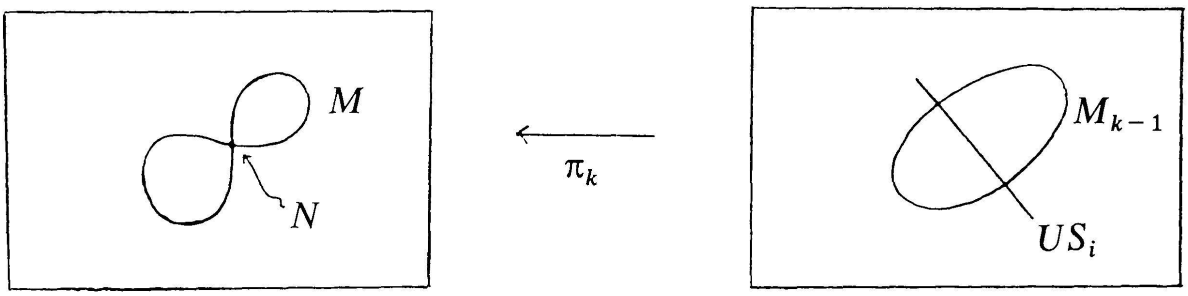
$$\pi_k^{-1}(V \times \mathbf{R}^m - N) = (V \times \mathbf{R}^m - N) \times \mathbf{R}^k .$$

Since M is an A_k-space, $\Sigma = \partial W$ for some compact A_{k-1}-space W. By proving a generalized version of Proposition 4.2 we can assume that the spine of W is a transversally intersecting codimension one A_{k-1} subspaces $\bigcup L_i$ with $\partial L_i = \emptyset$. We then imbed the A_{k-1} *space* $M_{k-1} = M_0 \cup N \times W$ (blown up M) into $B_k(V \times \mathbf{R}^m, N)$ such that

(i) M_{k-1} is transversal to $\bigcup S_i$ with $M_{k-1} \cap \bigcup S_i = N \times \bigcup L_i$,

(ii) $\pi_k(M_{k-1})$ is isotopic to M by a small isotopy,

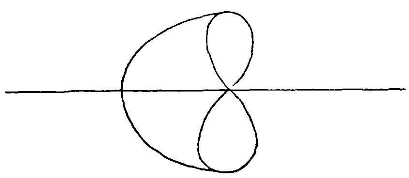
(iii) M_{k-1} represents 0 in $\mathcal{O}_*(B_k(V \times \mathbf{R}^m, N))$.



$V \times \mathbf{R}^m$ $B_k(V \times \mathbf{R}^m, N)$

This is somewhat hard to prove (see [AK_6]). Then by induction, with a small isotopy M_{k-1} can be moved to an algebraic subset Z of $B_k(V \times \mathbf{R}^m, N) \times \mathbf{R}^s$ for some s. Hence Z still satisfies (i) and (ii), after composing π_k with the obvious projection. Then by using a version of Proposition 3.3 we blow down Z to get an algebraic set homeomorphic to M.

The class of A-spaces does not contain all algebraic sets. For example the Whitney umbrella $x^2 = zy^2$ is not an A-space.



Therefore to classify real algebraic sets we need a bigger class of resolvable spaces (§6).

In order to show that P.L. manifolds admit A-structures one has to appeal to algebraic topological methods. This is done in [AT_2], here is a brief summary of

[AT_2]: One first verifies that A_k-structures on P.L. manifolds obey the usual structure axioms ([L]). For example they satisfy the product structure axiom i.e. for any P.L. manifold M an A_k-structure $(M \times I)_\gamma$ on $M \times I$ is concordant to $M_\gamma \times I$ where M_γ is an A_k-structure on M. Using [W] we can define an r-dimensional A_k-thickening on X to be a simple homotopy equivalence $X \xrightarrow{f} W^r$ where W^r is an r-dimensional A_k-manifold (with boundary). Let $T_k^r(X)$ to be the set of all r-dimensional A_k-thickenings on X with the equivalence relation: $(W_1, f_1) \sim (W_2, f_2)$ if there is an $(r+1)$-dimensional A_k-thickening (W, F) with $\partial W = W_1 \cup W_2$ and making the following diagram commute up to homotopy:

$$\begin{tikzcd} & W_1 \arrow[dr, hook] & \\ X \arrow[ur, "f_1"] \arrow[rr, "F"] \arrow[dr, "f_2"'] & & W \\ & W_2 \arrow[ur, hook] & \end{tikzcd}$$

There are natural maps $T_k^r(X) \to T_k^{r+1}(X)$ given by $(W, f) \mapsto (W \times I, f \times id)$, so using these maps we can take the direct limit $T_k(X) = \varinjlim T_k^r(X)$. It follows that the functor $X \mapsto T_k(X)$ is a representable functor (see [Sp]), hence by Brown representability theorem there exists a classifying space B_{A_k} such that $T_k(X) = [X; B_{A_k}]$. There are natural indusions $B_{A_{k-1}} \to B_{A_k}$, and let $B_A = \varinjlim B_{A_k}$. There is a natural forgetful map $B_A \xrightarrow{\pi} B_{PL}$. Then one shows that the usual structure theorem holds: Namely that a compact P.L. manifold M has an A-structure if and only if the normal bundle map (thickening map) $M \xrightarrow{\nu_M} B_{PL}$ lifts to B_A. Let PL/A be the homotopy theoretical fibre of π, then:

THEOREM 5.3 ([AT_2]). *$B_A \xrightarrow{\pi} B_{PL}$ is a trivial fibration, i.e. $B_A \simeq B_{PL} \times PL/A$ and PL/A is a product of Eilenberg-Mclain spaces $K(\mathbf{Z}/2\mathbf{Z}, n)$'s. The number ρ_n of $K(\mathbf{Z}/2\mathbf{Z}, n)$ for each n in this product is given by*

$$\rho_n = \begin{cases} 0 & \text{if } n < 8, \\ 26 & \text{if } n = 8, \\ \text{infinite but countable} & \text{if } n > 8. \end{cases}$$

COROLLARY 5.4. *Every compact P.L. manifold M has an A-structure and the number of different A-structures (up to A-concordance) on M is given by*

$$\bigoplus_{n \geq 8} H^n(M; \pi_n(PL/A)).$$

Briefly the proof of Theorem 5.3 goes as follows: By a standard argument, $\pi_i(PL/A_k)$ coincides with the concordance classes of A_k-structures on S^i (the exotic A_k-spheres). Since $\pi_i(PL/A) = \varinjlim \pi_i(PL/A_k)$ it follows by definitions that the inclusion $\pi_i(PL/A) \to \eta_i^A$ is an injection, where η_i^A is the cobordism group of i-dimensional A-manifolds. Then we construct a Thom space MA such that $\pi_i(MA) \approx \eta_i^A$ (by using a transversality argument for A-manifolds). Then it turns out that the map $\eta_i^A \to H_i(B_A; \mathbf{Z}/2\mathbf{Z})$ given by $\{M \xrightarrow{\nu_M} B_A\} \mapsto (\nu_M)_* [M]$ is an injection. We can put these maps into the following commutative diagram:

$$\begin{tikzcd}
& \pi_i(PL/A) \arrow[r] \arrow[dl, "h"'] \arrow[d, "f"] & \eta_i^A \arrow[d] \\
H_i(PL/A; \mathbf{Z}) \arrow[r, "r"] & H_i(PL/A; \mathbf{Z}/2\mathbf{Z}) \arrow[r, "g"] & H_i(B_A : \mathbf{Z}/2\mathbf{Z})
\end{tikzcd}$$

where h is the Hurewicz map, r is the reduction and g is induced by inclusion. Since the other two maps are injections then f must be injection. In fact f is a split injection since it is a map between $\mathbf{Z}/2\mathbf{Z}$-vector spaces. Hence h is a split injection. This implies that all k-invariants of PL/A is zero, i.e. PL/A is a product of Eilenberg-Mclaine spaces $\prod K(\mathbf{Z}/2\mathbf{Z}, n_i)$. Then by dualizing the split injection $g \circ f$ we get a surjection

$$H^i(B_A; \mathbf{Z}/2\mathbf{Z}) \xrightarrow{\lambda} \mathrm{Hom}(\pi_i(PL/A); \mathbf{Z}/2\mathbf{Z})$$

Let $\delta_{n_i} \in H^{n_i}(B_A; \mathbf{Z}/2\mathbf{Z})$ such that $\lambda(\delta_{n_i})$ is the generator of $\mathbf{Z}/2\mathbf{Z}$.

$$\delta = \prod \delta_{n_i} \text{ defines a map } B_A \to \prod_i K(\mathbf{Z}/2\mathbf{Z}, n_i) = PL/A\,.$$

Then the map $\pi \times \delta : B_A \to B_{PL} \times PL/A$ turns out to be the desired splitting. The calculation of ρ_n can be done by using the geometric interpretation of $\pi_*(PL/A)$.

The set $\mathscr{S}_A(M) = \bigoplus_n H^n(M; \pi_n(PL/A))$ measures the number of different "topological resolutions" of M, up to concordance (i.e. A-structures). Therefore often $\mathscr{S}_A(M)$ is infinite; and $\mathscr{S}_A(M^8)$ has 2^{26} elements for any closed 8-manifold M^8.

§6. On classification of Real Algebraic Sets

The resolution and complexification properties of real algebraic sets impose many restrictions on the underlying topological spaces. To give a topological characterization of algebraic sets one has to find all such properties, such that a