§1. Resolution of Algebraic Sets

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Consider $X_{\mathbf{C}} \subset \mathbf{CP}^n \times \mathbf{CP}^m$ and let $\pi_{\mathbf{C}} : X_{\mathbf{C}} \to V$ be the map induced by the projection to $\mathbf{CP}^m$ and $V = \pi_{\mathbf{C}}(X_{\mathbf{C}})$. By algebraic Sard's theorem (3.7 of [Mu]) $\pi_{\mathbf{C}}$ is a fibre bundle map over the complement of a complex algebraic subset W of V. The real part of W has real codimension $\geqq 1$ in $\overline{\pi(X)}$. Therefore if $\dim(\overline{\pi(X)} - \pi(X)) \geqq \dim \pi(X)$ then we can find a point $x_0 \in (\overline{\pi(X)} - \pi(X)) \cap (V - W)$. Also by hypothesis we can find a point $x_1 \in \pi(X) \cap (V - W)$ with $\chi(\pi^{-1}(x_1))$ odd. The sets $\pi_{\mathbf{C}}^{-1}(x_0)$ and $\pi_{\mathbf{C}}^{-1}(x_1)$ are invariant by complex conjugation, and the fixed point sets of the involutions induced by the complex conjugation are the empty set and $\pi^{-1}(x_1)$, respectively. Hence $\chi(\pi_{\mathbf{C}}^{-1}(x_0)) = 0 \pmod 2$ and

$$\chi(\pi_{\mathbf{C}}^{-1}(x_1)) = \chi(\pi^{-1}(x_1)) = 1 \pmod 2 ;$$

this is a contradiction since $\pi_{\mathbf{C}}^{-1}(x_0) \approx \pi_{\mathbf{C}}^{-1}(x_1)$ (because π is a fibre bundle map over $V - W$). □

§1. Resolution of Algebraic Sets

Another important property of algebraic sets is the resolution property. This property forces algebraic sets to satisfy many topological conditions (see §5). Given an algebraic set V and an algebraic subset L; the *algebraic blowup of V along L* $B(V, L)$ defined to be the Zariski closure of

$$\{(x, \theta\, f(x)) \in V \times \mathbf{RP}^{n-1} \mid x \in V - L\} ,$$

where $f : (V, L) \to (\mathbf{R}^n, 0)$ is a polynomial whose coordinates generate $I(L)/I(V)$ and $\theta : \mathbf{R}^n - \{0\} \to \mathbf{RP}^{n-1}$ is the quotient map $\theta(x_1, \ldots, x_n) = [x_1 : \ldots : x_n]$. The amusing fact is that $B(V, L)$ is well defined algebraic subset of $V \times \mathbf{RP}^{n-1}$. Furthermore if V and L are nonsingular then $B(V, L)$ is diffeomorphic to the topological blowup of V along L $B_t(V, L) = (V - \text{interior } N) \cup E(N)$ where N is the normal disc bundle of L in V and $E(N)$ is the I-bundle over the projectivized normal bundle of L in V, i.e. $E(N)$ is obtained by replacing each fiber D^k of N by $\mathbf{RP}^k - \operatorname{int}(D^k)$. There are natural projections π, π_t making the following commute

$$\begin{tikzcd} B(V, L) \arrow[dr, "\pi"] \arrow[d, phantom, "\wr\wr"] & \\ B_t(V, L) \arrow[r, "\pi_t"'] & V \end{tikzcd}$$

Given any polyhedron M with $L \subset M \subset V$ where L, V smooth manifolds then we define $B_t(M, L)$ to be the closure of $\pi_t^{-1}(M) - \pi_t^{-1}(L)$ in $B_t(V, L)$.

If M is a smooth manifold this definition coincides with the usual $B_t(M, L)$. From now on we drop the subscript and let $B(M, L) \xrightarrow{\pi} M$ to denote the topological (algebraic) blowup if $L \subset M$ are manifolds (algebraic sets). Any inclusions $L \subset M \subset V$ give rise to inclusions $B(M, L) \subset B(V, L)$. Given smooth manifolds $L \subset M \subset V$ and $B(V, L) \xrightarrow{\pi} V$ then $\pi^{-1}(L)$ is the projectivized normal bundle $P(L, V)$ of L in V and $\pi^{-1}(L) \cap B(M, L) = P(L, M)$.

Let V be a nonsingular algebraic set (a smooth manifold) and M be an algebraic subset (a smooth stratified subset). Then $\tilde{V} \xrightarrow{\pi} V$ is called an *algebraic (topological) multiblowup* of V along M if: $\pi = \pi_1 \circ \pi_2 \circ \ldots \circ \pi_k$ for some k, where $\tilde{V} = V_k \xrightarrow{\pi_k} V_{k-1} \xrightarrow{\pi_{k-1}} \ldots \xrightarrow{\pi_1} V_0 = V$ such that $V_{i+1} = B(V_i, L_i) \xrightarrow{\pi_i} V_i$ are blowups along nonsingular algebraic subsets (closed smooth submanifolds) L_i of V_i. Furthermore $L_i \subset M_i$ with $\dim(L_i) < \dim(M_i)$ where $M_{i+i} = B(M_i, L_i)$, $M_0 = M$, and M_k is a nonsingular algebraic subset (a smooth submanifold) of V_k. We will denote M_k by $\tilde{M}$. $\tilde{M}$ is usually called the *strict preimage* of M and L_i's are called the *centers* of the multiblowup. If furthermore the imbeddings $L_i \subset V_i$ and $\tilde{M} \subset \tilde{V}$ satisfy some particular property $\mathscr{P}$ we call $\tilde{V} \xrightarrow{\pi} V$ a *$\mathscr{P}$ algebraic (topological) multiblowup.*

Notice that if $V \subset \mathbf{R}^n$ is an algebraic set then we can assume that

$$\tilde{V} \subset \mathbf{R}^n \times \prod_{i=1}^{k} \mathbf{RP}^{a_i} \subset \mathbf{R}^n \times \mathbf{R}^m$$

for some m and $\pi: \tilde{V} \to V$ is induced by the projection $\mathbf{R}^n \times \mathbf{R}^m \to \mathbf{R}^m$.

THEOREM 1.1 (Hironaka [H]). *Let V be a nonsingular algebraic set and M be an algebraic subset. Then there exists an algebraic multiblowup $\tilde{V} \xrightarrow{\pi} V$ along M. Furthermore $\pi|_{\pi^{-1}(\text{Non sing } M)}$ is a birational diffeomorphism.*

This theorem says that a singular algebraic set can be made nice (nonsingular) by blowing up along nice (nonsingular) algebraic subsets. We can go one step further, namely starting with a nonsingular algebraic set we can make it nicer (fine) by blowing up along nicer (fine) algebraic subsets. First we need some definitions: Let $M \subset V$ be nonsingular algebraic sets, then M is called a *fine algebraic subset* if it is a component of a transversally intersecting codimension one compact nonsingular algebraic subset of V. M is called a *stable algebraic subset* if $M = Z_0 \subset Z_1 \subset \ldots \subset Z_{r+1} = V$ where $\{Z_i\}_{i=0}^r$ are compact nonsingular algebraic subsets with $\dim(Z_{i+1}) = \dim(Z_i) + 1$. Similarly in these definitions by changing nonsingular algebraic sets with smooth manifolds we define *fine submanifolds* and *stable submanifolds*.

Clearly fine algebraic subsets (submanifolds) are stable algebraic subsets (submanifolds). Stable algebraic subsets are useful because they obey transversality (Theorem 2.7). In algebraic geometry sometimes fine algebraic subsets are called complete intersections. If M is compact and has a trivial normal bundle in V then M is a fine submanifold of V.

THEOREM 1.2 ([AK_8]). *Let V be a nonsingular algebraic set and M be a compact nonsingular algebraic subset. Then there exists a fine algebraic multiblowup $\tilde{V} \xrightarrow{\pi} V$ along M.*

Since any pair of closed smooth manifolds $M \subset V$ are pairwise diffeomorphic to nonsingular algebraic sets (Theorem 2.12), Theorem 1.2 has the obvious topological version. An application of this theorem is Proposition 2.11 (the definition of $\sigma(\theta)$).

There is a homology version of the resolution theorem, which says that $\mathbf{Z}/2Z$-cocycles (or cycles) can be desingularized by blowing up. For a given compact nonsingular algebraic set V let $H_*^A(V; \mathbf{Z}/2\mathbf{Z})$, $AH_*(V; \mathbf{Z}/2\mathbf{Z})$, $H_*^{imb}(V; \mathbf{Z}/2\mathbf{Z})$ denote the subgroups of $H_*(V; \mathbf{Z}/2\mathbf{Z})$ generated by algebraic subsets, stable algebraic subsets, imbedded closed smooth submanifolds respectively. Let $H_A^*(V; \mathbf{Z}/2\mathbf{Z})$, $AH^*(V; \mathbf{Z})$, $H_{imb}^*(V; \mathbf{Z}/2\mathbf{Z})$ denote the Poincaré duals of these subgroups.

THEOREM 1.3 ([AK_8]). *Let V be a compact nonsingular algebraic set, then there exists an algebraic multiblowup $\tilde{V} \xrightarrow{\pi} V$ such that, for all i*

(a) $\pi^* H^i(V; \mathbf{Z}/2\mathbf{Z}) \subset H_{imb}^i(\tilde{V}; \mathbf{Z}/2\mathbf{Z})$

(b) $\pi^* H_A^i(V; \mathbf{Z}/2\mathbf{Z}) \subset AH^i(\tilde{V}; \mathbf{Z}/2\mathbf{Z})$

Furthermore if we fix i we can assume that the centers of the multiblowup has dimension $< \dim(V) - i$.

As a corollary to the proof of Theorem 1.3 one gets an algebraic version of Steenrod representability theorem:

COROLLARY 1.4. *If V is a nonsingular algebraic set and*

$$\theta \in H_k(V; \mathbf{Z}/2\mathbf{Z}),$$

then there exists an algebraic multiblowup $\tilde{V} \xrightarrow{\pi} V$ along the centers of dimension less than k and a k-dimensional nonsingular algebraic subset Z of $\tilde{V}$ and a component Z_0 of Z, such that $\pi|_{Z_0}: Z_0 \to V$ represents θ.

(b) Implies that the algebraic cohomology $H^*_A(V; \mathbf{Z}/2\mathbf{Z})$ is closed under cohomology operations [AK₈]. For example to show that the intersection of two algebraic homology classes is an algebraic homology class we take a resolution $\tilde{V} \xrightarrow{\pi} V$ which makes these algebraic subsets stable algebraic subsets, then by Theorem 2.7 we can make them transversal and project the intersection back into V, then the Zariski closure of this set corresponds to the homology intersection of the original homology classes.

Since any closed smooth manifold is diffeomorphic to a nonsingular algebraic set (a) applies to smooth manifolds. It gives some interesting topological corollaries. Here is an example: Let $MO(r)$ be the Thom space [T] of the universal $\mathbf{R}^r$-bundle. The Thom class generates

$$H^r(MO(r) : \mathbf{Z}/2\mathbf{Z}) \cong H^{r+n}(\Sigma^n MO(r) ; \mathbf{Z}/2\mathbf{Z})$$

hence it defines a map $\Sigma^n MO(r) \to K(\mathbf{Z}/2\mathbf{Z}, r+n)$. By taking n-fold loops on both sides we get a natural map

$$p : \Omega^n \Sigma^n MO(r) \to K(\mathbf{Z}/2\mathbf{Z}, r) .$$

It is well known that any r-dimensional cohomology class of a closed smooth manifold M is classified by a map $f : M \to K(\mathbf{Z}/2\mathbf{Z}, r)$ and the dual of this cohomology class can be represented by an immersed submanifold if and only if f lifts to $\Omega^n\Sigma^n MO(r)$ for some large n. So it is useful to understand the map p. Interestingly, Theorem 1.3 implies that p is an injection in $\mathbf{Z}/2\mathbf{Z}$ cohomology as follows: By taking the boundary of a tubular neighborhood V of some big skeleton of $K(\mathbf{Z}/2\mathbf{Z}, r)$ in $\mathbf{R}^n$ we get an inclusion $f : V \to K(\mathbf{Z}/2\mathbf{Z}, r)$ with f^* isomorphism for large $*$. By Theorem 1.3 we can take a multiblowup $\tilde{V} \xrightarrow{\pi} V$ with $\pi^* f^*(\iota) \in H^*_{imb}(\tilde{V} : \mathbf{Z}/2\mathbf{Z})$, where ι is the fundamental class. Hence the dual of $\pi^* f^*(\iota)$ is represented by an immersed submanifold, therefore there is a map g making the following commute

$$\begin{array}{ccc} \tilde{V} & \xrightarrow{g} & \Omega^n\Sigma^n MO(r) \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle p} \\ V & \xrightarrow{f} & K(\mathbf{Z}/2\mathbf{Z}, r) \end{array}$$

Since π is a degree 1 map it is an injection in cohomology, hence p^* must be an injection.