Appendix 1 Relations between polylogarithm and Hurwitz function

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is non-zero according to Dirichlet. Thus we obtain a contribution of $-1 + \varphi(m)/2$ to the rank coming from the non-trivial even characters.

On the other hand, for the eigenspace corresponding to the trivial character, using formula (10) of §4 we obtain a contribution equal to the number of primes dividing m. Lemmas 8 and 10 of §5 now complete the proof. □

Appendix 1

Relations between polylogarithm and Hurwitz function

For every complex number s, it follows from Theorem 1 that there exists a linear relation between the even [or the odd] part of the function $l_s(x)$ and of the function $\zeta_{1-s}(x)$ or $\beta_s(x) = -s\zeta_{1-s}(x)$. This appendix will work out the precise form of these relations. Compare [3], [19], [27].

For integer values of s, the required relation can be obtained as follows. Recall from formula (9) of §2 that

$$l_0(x) = (-1+i \cot \pi x)/2$$

hence

$$l_0(x) + l_0(1-x) + \beta_0(x) = 0 .$$

Integrating, we see that

$$l_1(x) - l_1(1-x) + 2\pi i\, \beta_1(x)/1! = 0$$

$$l_2(x) + l_2(1-x) + (2\pi i)^2 \beta_2(x)/2! = 0$$

and so on, for $0 < x < 1$. For even values of the subscript, specializing to $x = 0$ as in §4, this yields Euler's formula

$$2\zeta(2k) + (2\pi i)^{2k} b_{2k}/(2k)! = 0 .$$

In particular, it follows that $\zeta(0) = -\dfrac{1}{2}$, and that the numbers $b_2, -b_4, b_6, -b_8, \ldots$ are strictly positive. On the other hand, differentiating the formula for $l_0(x)$, we obtain

$$l_{-1}(x) = -csc^2(\pi x)/4 .$$

This is an even function satisfying $(*_{-1})$, so it must be some multiple of $\zeta_2(x) + \zeta_2(1-x)$. Comparing asymptotic behavior as $x \to 0$, we obtain the classical formula

$$\zeta_2(x) + \zeta_2(1-x) = \pi^2/\sin^2 \pi x = (2\pi i)^2 l_{-1}(x)/1! .$$

Differentiating, we see that

$$-\zeta_3(x) + \zeta_3(1-x) = (2\pi i)^3 l_{-2}(x)/2!$$
$$\zeta_4(x) + \zeta_4(1-x) = (2\pi i)^4 l_{-3}(x)/3!$$

and so on.

For $s \neq 0, 1, 2, \ldots$ we know from §3 that there is some relation of the form

$$l_s(x) = A_s \zeta_{1-s}(x) + B_s \zeta_{1-s}(1-x) \tag{14}$$

for $0 < x < 1$; where A_s and B_s are certain uniquely determined constants. Expressing each of these functions of x as the sum of an even part and an odd part, we see that

$$\begin{cases} l_s^{\text{even}}(x) = (A_s+B_s)\zeta_{1-s}^{\text{even}}(x) \\ l_s^{\text{odd}}(x) = (A_s-B_s)\zeta_{1-s}^{\text{odd}}(x) \, . \end{cases} \tag{15}$$

Evidently the functions $s \mapsto A_s \pm B_s$ *are meromorphic, taking finite non-zero values for all* $s \in \mathbf{C} - \mathbf{Z}$. Differentiating with respect to x, we see that

$$A_s \pm B_s = s(A_{s+1} \mp B_{s+1})/(2\pi i) \, . \tag{16}$$

For integral values of s, using the discussion above, we easily obtain the following table of values, where $0! = 1$.

s	...	-2	-1	0	1	2	3	...
$A_s + B_s$	...	0	$\frac{2 \cdot 1!}{(2\pi i)^2}$	0	∞	$\frac{(2\pi i)^2}{2 \cdot 1!}$	∞	...
$A_s - B_s$	...	$-\frac{2 \cdot 2!}{(2\pi i)^3}$	0	$-\frac{2 \cdot 0!}{2\pi i}$	$\frac{2\pi i}{2 \cdot 0!}$	∞	$\frac{(2\pi i)^3}{2 \cdot 2!}$	...

Now suppose that we specialize to $x = 0$, by the procedure of §4. Then equation (14) reduces to a form

$$\zeta(s) = (A_s+B_s)\zeta(1-s)$$

of Riemann's functional equation. It follows that

$$(A_s+B_s)(A_{1-s}+B_{1-s}) = 1 \, ,$$

and hence using (16) that

$$(A_s-B_s)(A_{1-s}-B_{1-s}) = -1 \, .$$

This discussion gives all of the information about $A_s \pm B_s$ which we will need. However, it is possible to compute precise values as follows. Let $\zeta_{1-s}(e^{2\pi i}x)$ be the result of analytic continuation in a loop circling the origin. Then evidently

$$\zeta_{1-s}(e^{2\pi i}x) - \zeta_{1-s}(x) = (e^{2\pi i s} - 1)x^{s-1}.$$

Using the integral formula (6), computation shows that

$$l_s(e^{2\pi i}x) - l_s(x) = -(2\pi i)^s x^{s-1}/\Gamma(s).$$

Comparing these two expressions, and noting that $\zeta_{1-s}(1-x)$ is holomorphic throughout a neighborhood of $x = 0$, we can solve for A_s. The result after some manipulation is

$$A_s = \frac{i(2\pi)^s e^{-\pi i s/2}}{2\Gamma(s)\sin(\pi s)}.$$

Now comparing the behavior of l_s and ζ_{1-s} under complex conjugation we see easily that

$$B_s = \overline{A_{\bar{s}}} = \frac{-i(2\pi)^s e^{\pi i s/2}}{2\Gamma(s)\sin(\pi s)}.$$

In particular, it follows that

$$A_s + B_s = \frac{(2\pi)^s}{2\Gamma(s)\cos(\pi s/2)}, \qquad A_s - B_s = \frac{i(2\pi)^s}{2\Gamma(s)\sin(\pi s/2)}.$$

As an application of formula (15), let us prove the corresponding functional equation for a Dirichlet L-function. Recall from Lemma 14 that for any primitive Dirichlet character χ modulo m the function

$$L(s, \chi) = \sum_1^m \chi(k)\zeta_s(k/m)/m^s$$

satisfies

$$L(s, \bar{\chi}) = \sum_1^m \chi(k) l_s(k/m)/\tau.$$

Here we may just as well use either the even or the odd parts of ζ_s and l_s according as $\chi(-1)$ is $+1$ or -1. Therefore, it follows from (15) that

$$\begin{aligned} L(s, \bar{\chi}) &= (A_s \pm B_s)\sum_1^m \chi(k)\zeta_{1-s}(k/m)/\tau \\ &= m^s(A_s \pm B_s)L(1-s, \chi)/\tau. \end{aligned}$$

Thus we have proved the functional equation

(17) $$L(s, \bar{\chi}) = m^{1-s}(A_s + \chi(-1)B_s)L(1-s, \chi)/\tau(\chi).$$

Here the factor m^{1-s}/τ is never zero or infinite, while $A_s \pm B_s$ is zero or infinite only at certain integer values, as indicated in the table above.

The proof of Lemma 13 can now easily be completed as follows. If $s \leqslant 0$ is an integer, then $L(1-s, \chi) \neq 0$, so it follows that $L(s, \bar{\chi})$ equals zero if and only if $A_s \pm B_s$ is zero, as indicated in the table. □

Appendix 2

Some relatives of the gamma function

This appendix will describe certain functions $\gamma_1(x)$, $\gamma_2(x)$, ... which satisfy a modified form of the Kubert identities, with a polynomial correction term. (See (22) below.) They are defined as partial derivatives of the Hurwitz function by the formula

$$\gamma_{1-t}(x) = \partial\zeta_t(x)/\partial t \,. \tag{18}$$

We will show that γ_1 is related to the classical gamma function via Lerch's identity

$$\gamma_1(x) = \log(\Gamma(x)/\sqrt{2\pi}) \,. \tag{19}$$

(Compare [27, p. 60].) As a bonus, we will give a self-contained exposition of the basic properties of the gamma function, based on formulas (18) and (19).

Let us begin with equation (18), which defines $\gamma_s(x)$ as an analytic function of both variables for all $s \neq 0$ and all $x > 0$. Recall that the Hurwitz function $\zeta_t(x) = x^{-t} + (x+1)^{-t} + \ldots$ (analytically extended in t for $t \neq 1$) satisfies

$$\zeta_t(x+1) = \zeta_t(x) - x^{-t} \,.$$

Differentiating with respect to t, and then substituting $t = 1 - s$, we obtain

$$\gamma_s(x+1) = \gamma_s(x) + x^{s-1} \log x \,. \tag{20}$$

In particular,

$$\gamma_1(x+1) = \gamma_1(x) + \log x \,.$$

Note that

$$\zeta_t'(x) = -t\zeta_{t+1}(x)$$

hence

$$\zeta_t''(x) = t(t+1)\zeta_{t+2}(x) \,,$$