4. Closure properties

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Suppose now that a family P is p-definable in the sense of Definition 4. Then the argument in Proposition 1 showing that Definition 3 implies Definition 2 establishes that P is p-definable in the sense of Definition 1*. But Theorem 3 in [13] shows that any P so definable is the p-projection of HC and our Appendix 2 shows that HC is p-definable in the sense of Definition 1. The result follows. □

In Appendix 1 it will be shown that Definition 3 implies Definition 4. Together with Propositions 1 and 2 this will establish:

THEOREM 1. *Definitions 1, 2, 3 and 4 are all equivalent.*

4. CLOSURE PROPERTIES

A p-definable family P is *complete* over F if every family that is p-definable over F is the p-projection of P. It is known that several famous polynomials such as the permanent, hamiltonian circuits, the monomer-dimer polynomial and certain reliability problems are all complete for appropriate fields [6, 13]. In fact the projections required to establish these facts are all *strict projections* (i.e. no two indeterminates map to the same indeterminate). Hence these superficially dissimilar polynomials are related in the closest possible way: each one can be obtained from any other by fixing some indeterminates and renaming the others.

In the light of the simplicity of its completeness class the robustness of the notion of p-definability is perhaps remarkable. It can be explored conveniently by listing the operations under which it is closed.

First we consider the operation of substitution. The polynomials to be substituted can be viewed conveniently as an array.

Definitions. R is a *family array* over F if it is a set $\{ R^{m,n} \mid n \leqslant m \}$ of polynomials over F where $R^{m,n}$ has m indeterminates. It has *p-bounded degree* if for some p-bounded t $\deg(R^{m,n}) \leqslant t(m)$.

The various definitions of p-definability have analogues that are equivalent to each other for family arrays. For the current purpose it is best to adapt the fourth one:

Definition. Family array R is *p-definable* iff there is a p-bounded t such that for all m, n there is a T with formula size less than $t(m)$ such that

$$R^{m,n} = \sum_{\mathbf{b}} T(\mathbf{x}, \mathbf{b}) .$$

THEOREM 2. *If family P and array R are p-definable over F then so is the family $P(R) = \{P_m(R^{m,1}, R^{m,2}, \ldots, R^{m,m})\}$*

Proof. Consider the two polynomials:

$$S_i^1(\mathbf{x}) = \sum_{\mathbf{b}} Q^1(\mathbf{x}, b_1, \ldots, b_k) \quad and \quad S_j^2(\mathbf{y}) = \sum_{\mathbf{c}} Q^2(\mathbf{y}, c_1, \ldots, c_r)\,.$$

If $k \geqslant r$ then their product is

$$\sum_{\mathbf{b}} \sum_{\mathbf{c}} Q^1(\mathbf{x}, b_1, \ldots, b_k) \cdot Q^2(\mathbf{y}, c_1, \ldots, c_r)$$

and their sum

$$\sum_{\mathbf{b}} Q^1(\mathbf{x}, b_1, \ldots, b_k) + Q^2(\mathbf{y}, b_1, \ldots, b_r)\, b_{r+1} \ldots b_k\,.$$

It follows by induction on the construction of formulae that if S is any family with p-bounded formula size then $S(R)$ is p-definable. Now choose S to be the family defining P. A typical member of $P(R)$ is

$$P_m(R) = \sum_{\mathbf{d}} S(R^{m,1}, \ldots, R^{m,m}, \mathbf{d})\,.$$

It follows by Theorem 5 that $P(R)$ is also p-definable. □

Remark 2. Closure of p-definability under addition ensures that Perm + 1 is p-definable. Since Perm is complete it follows that Perm + 1 is the p-projection of Perm. No direct proof of this is known and it is noteworthy that the corresponding question as to whether Det + 1 is the p-projection of Det appears to be open.

Remark 3. Reliability polynomials such as those considered in [6] can be recognised as p-definable by first considering distinct indeterminates p, q for each edge, and then substituting $q = 1 - p$.

The *coefficient* in $P_n \in F[x_1, \ldots, x_n]$ of the monomial $m = x_1^{i_1} \ldots x_n^{i_n}$ is the unique polynomial Q_n such that (i) $P_n = mQ_n + R_n$, (ii) Q_n and m have no indeterminate in common, and (iii) each monomial in R_n differs from m in the exponent of at least one indeterminate occurring in m.

The following closure property strengthens Proposition 9 in [13].

THEOREM 3. *If P is p-definable and R is a family such that for some p-bounded t, for each i, R_i is a coefficient in $P_{t(i)}$ then R is p-definable also.*

Proof. Suppose that $P_{t(i)}$ is the projection of

$$U = \sum_{\mathbf{b}} Q_j(\mathbf{b}) \prod_{b_k = 1} x_k$$

under σ. If R_i is the coefficient of $m = \prod y_s^{i_s}$ in $P_{t(i)}$ then it is the projection under σ of the sum of the coefficients in U of all products $\prod_{b_k=1} x_k$ such that for each s with $i_s \geqslant 1$.

$$|\{k \mid b_k = 1 \quad \text{and} \quad \sigma(x_k) = y_s\}| = i_s\,.$$

It therefore follows that R_i is a projection under σ' of

$$\sum_{\mathbf{b}} Q_j(\mathbf{b}) \prod_{s=1} \mathrm{Sym}^{i_s}(b_r \mid \sigma(x_r) = y_s) \prod_{b_k=1} x_k \;,$$

where Sym is the polynomial defined in §2, and σ' modifies σ by mapping each element of

$$\{x_k \mid \sigma(x_k) = y_s \quad \text{and} \quad i_s \geqslant 1\}$$

to unity. □

THEOREM 4. *If* P *is p-definable then so are*

(i) $\{\partial P_i / \partial x_j \mid P_i \in P, \quad \text{any} \quad j\}$,

(ii) $\{\int P_i \, dx_j \mid P_i \in P, \quad \text{any} \quad j\}$, and

(iii) *the result of any p-bounded number of applications to* P *of differentiation or integration.*

Proof (i). Suppose that P_i is the projection of

$$\sum_{\mathbf{b}} Q_n(\mathbf{b}) \prod_{b_k=1} y_k$$

under $\sigma : \{y_k\} \to \{x_m\} \cup F$. For each power x_j^q of x_j we will take its coefficient, multiply it by $q z_1 \dots z_{q-1}$ where $z_1, \dots, z_{q-1}$ are new indeterminates, and finally project the original x_j to one and the new z's to x_j. Let $S = S_1 + S_2 + \dots + S_d$ where $d = \deg(P_i)$ and $S_q(\mathbf{b}, \mathbf{c})$ equals:

$$q \cdot \mathrm{Sym}^q(b_r \mid \sigma(y_r) = x_j) \cdot \mathrm{Sym}_{q-1}^{q-1}(c_1, \dots, c_{q-1}) \cdot \mathrm{Sym}_{d-q+1}^{0}(c_q, \dots, c_d)$$

Then $\partial P_i / \partial x_j$ is the projection of

$$\sum Q_n(\mathbf{b})\, S(\mathbf{b}, \mathbf{c}) \prod_{b_k=1} y_k \prod_{c_s=1} z_s \;.$$

Parts (ii) and (iii) follow by similar arguments. □

Finally we note that while p-definable families are rich in closure properties the p-computable ones are apparently not. Numerous natural mathematical operations seem not to preserve tractability. We can explore this phenomenon formally by showing that some easy polynomials become complete when so operated on. A most convenient starting point is the following family T which is of p-bounded formula size:

$$T_{n^2+n} = \prod_{k=1}^{n} \sum_{i=1}^{n} x_{k,i} \, y_i \; .$$

Clearly (i) the coefficient of $y_1 \dots y_n$ in T_{n^2+n},

(ii) $$\frac{\partial}{\partial y_1} \frac{\partial}{\partial y_2} \cdots \frac{\partial}{\partial y_n} T_{n^2+n} \, , \text{ and}$$

(iii) $$\left(\frac{3}{2}\right)^n \int_{-1}^{1} \cdots \int_{-1}^{1} [y_1 \dots y_n \, T_{n^2+n}] \, dy_1 \dots dy_n$$

all equal Perm $\{ x_{k,i} \}$.

In contrast, it is easy to see that all the other operations that we have considered preserve p-computability. This is immediate in the case of substitution. It can be shown to be true for $\partial P/\partial x_i$ and $\int P dx_i$ by considering a program for P, and decomposing it according to the powers of x_i at each instruction in the manner of [12].

5. A Non-existent Hierarchy

By analogies with recursion theory we can attempt to define the following hierarchy:

Definition. PD^0 = class of p-computable polynomial families. For $i > 0$ $P \in PD^i$ iff P is defined by some $Q \in PD^{i-1}$ in the sense of Definition 3.

That this hierarchy collapses in this algebraic case is easy to see:

THEOREM 5. *For any* F *and any* $i > 0$ $PD^i = PD^{i+1}$.

Proof. It is clearly sufficient to prove $PD^1 = PD^2$. If $P \in PD^2$ then for each m

$$P_m(\mathbf{x}) = \sum_{\mathbf{b}} Q_i(\mathbf{x}, \mathbf{b})$$

where for some $R \in PD^0$ for each i

$$Q_i(\mathbf{x}, \mathbf{b}) = \sum_{\mathbf{c}} R_j(\mathbf{x}, \mathbf{b}, \mathbf{c}) \; .$$

Hence

$$P_m(x) = \sum_{\mathbf{b}, \mathbf{c}} R_j(\mathbf{x}, \mathbf{b}, \mathbf{c})$$

which shows that $P \in PD^1$. □