4. V-BOUNDEDNESS, WEAK AND STRONG HARMONIZABILITY

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$F : \mathbf{R} \times \mathbf{R} \to \mathbf{C}$ of locally finite N-variation (let N' be the associate norm of N), and there exists a family $\{g_t, t \in \mathbf{R}\}$ of Borel functions which are MT-integrable relative to F, such that

$$r(s, t) = \int_{\mathbf{R}} \int_{\mathbf{R}} g_s(\lambda)\bar{g}_t(\lambda')F(d\lambda, d\lambda'), \; s, t \in \mathbf{R}\,, \tag{33}$$

and where locally finite N-variation is meant the following:

$$\infty > \| F \|_N(A \times A) = \sup \{ \, | I(f, g) | : N'(f) \leqslant 1, N'(g) \leqslant 1 \} \,. \tag{34}$$

Here f, g are Borel step functions, with $\mathrm{supp}(f) \subset A$, $\mathrm{supp}(g) \subset A$, $A \in \mathscr{B}_0$, the δ-ring of bounded Borel sets of $\mathbf{R}$.

(b) A process $X : \mathbf{R} \to L_0^2(P)$ is of $\mathrm{class}_N(\mathrm{C})$ if its covariance function r is of class_N (C) so that it is representable as (33).

It is clear that if $N(\cdot) = \| \cdot \|_1$ so that $N'(\cdot) = \| \cdot \|_\infty$, the N-variation is simply the 1-semivariation of Definition 3.1 and that

$$\| F \|_N = \| F \|_1 \, (= \| F \|) \,.$$

Remark. Without further restrictions, class_N (C) need not contain the weak or strong harmonizable processes. However if N is restricted so that, letting

$$L^N(P) = \{f \in M(P) : N(f) < \infty\}, L^\infty(P) \subset L^N(P) \subset L^1(P)\,,$$

where $\mu = P$ is a probability, then every class_N (C) will contain both the weak and strong harmonizable families, as an easy computation shows. If $N(\cdot) = \| \cdot \|_1$, then class_1 (C) is the class which corresponds to the covariance bimeasure of *finite semivariation.* This includes the classical Loève and Rozanov definitions. Again this definition holds, with only a notational change, if $\mathbf{R}$ is replaced by a locally compact group G. A brief discussion on some analysis of these classes which extend the present work is included at the end of the paper.

4. V-BOUNDEDNESS, WEAK AND STRONG HARMONIZABILITY

The definition of weak harmonizability is of interest only when an effective characterization of it is found and when its relations with strong harmonizability are made concrete. These points will be clarified and answered here. Now Theorem 3.3 shows that a weakly harmonizable process is the Fourier transform of a stochastic measure and this leads to a fundamental concept called V-boundedness ('V' for "variation"), introduced much earlier by Bochner [2], which is valid in a more general context. This notion plays a central role in the theory and applications of weakly harmonizable processes (and fields) which are shown to be V-bounded in the context of $L_0^2(P)$. Further this characterization facilitates a use of the powerful tools of Fourier analysis of vector measures. The desired concept is as follows (cf. [2], and also [33]):

Definition 4.1. A process $X : \mathbf{R} \to \mathscr{X}$, a Banach space, is *V-bounded* if $X(\mathbf{R})$ lies in a ball of $\mathscr{X}$, X as an $\mathscr{X}$-valued function is strongly measurable (i.e., range of X is separable and $X^{-1}(B) \in \mathscr{B}$ for each Borel set $B \subset \mathscr{X}$), and if the set C is relatively weakly compact in $\mathscr{X}$, where

$$C = \{\textstyle\int_{\mathbf{R}} f(t)X(t)dt : \| \hat{f} \|_u \leqslant 1, f \in L^1(\mathbf{R})\} \subset \mathscr{X}, \tag{35}$$

and where $\hat{f}(t) = \int_{\mathbf{R}} f(\lambda)e^{it\lambda}d\lambda$, $\int_{\mathbf{R}} f(t)X(t)dt$ being the Bochner integral. If $\mathscr{X}$ is reflexive then the condition on C may be replaced by its boundedness. (Here if the measurability of X is strengthened to weak continuity, then it actually implies the strong [and even uniform] continuity.)

Let us establish the following basic fact when $\mathscr{X} = L_0^2(P)$:

THEOREM 4.2. *A process $X : \mathbf{R} \to L_0^2(P)$ is weakly harmonizable iff X is V-bounded (i.e., $\| X(t) \|_2 \leqslant M_0 < \infty, t \in \mathbf{R}$, and the set in (35) is bounded) and weakly continuous.*

Proof: For the direct part, let X be weakly continuous and V-bounded. Then

$$\| \textstyle\int_{\mathbf{R}} f(t)X(t)dt \|_2 \leqslant c \| \hat{f} \|_u, f \in L^1(\mathbf{R}), \tag{36}$$

by Definition 4.1. Let $\mathscr{Y} = \{\hat{f} : f \in L^1(\mathbf{R})\} \subset C_0(\mathbf{R})$, the space of complex continuous functions vanishing at "∞"; the inclusion holds by the Riemann-Lebesgue lemma. Moreover, $\mathscr{Y}$ is uniformly dense in $C_0(\mathbf{R})$, since $\mathscr{Y}$ is a real algebra in $C_0(\mathbf{R})$ and separates points of $\mathbf{R}$ so that the Stone-Weierstrass theorem applies (cf. [24], §26.B). Let $\mathscr{F} : f \mapsto \int_{\mathbf{R}} f(\lambda)\bar{e}_t(\lambda)d\lambda$, $t \in \mathbf{R}$, where $e_t(\lambda) = e^{it\lambda}$. Then $\mathscr{F} : L^1(\mathbf{R}) \to C_0(\mathbf{R})$ is a one-to-one contractive operator. Consider the mapping

$$T : \mathscr{Y} \to \mathscr{X} = L_0^2(P), \quad \text{by} \quad T(\hat{f}) = \textstyle\int_{\mathbf{R}} f(t)X(t)dt \in \mathscr{X} .$$

This is well-defined, and the following diagram is commutative:

$$T_1(f) = \textstyle\int_{\mathbf{R}} f(t)X(t)dt \in \mathscr{X} .$$

$$\begin{tikzcd} L^1(\mathbf{R}) \arrow[rr, "\mathscr{F}"] \arrow[dr, "T_1"'] & & \mathscr{Y} \arrow[dl, "T"] \\ & \mathscr{X} & \end{tikzcd}$$

By hypothesis T is bounded and by the density of $\mathscr{Y}$ in $C_0(\mathbf{R})$, it has a norm preserving extension $\tilde{T}$ to $C_0(\mathbf{R})$. Now $\tilde{T}$ will be given an integral representation using a classical theorem due to Dunford-Schwartz ([8], VI.7.3) since $\tilde{T}$ is a weakly compact operator because $\mathscr{X}$ is reflexive.

To invoke the above cited theorem, however, it should first be observed that the result holds even if the space $C(S)$ of continuous (scalar) functions on a compact space S (for which it is proved) is replaced by $C_0(\mathscr{S})$ with a locally compact space $\mathscr{S}$. Here $\mathscr{S} = \mathbf{R}$. Indeed, let $\bar{\mathscr{S}}$ be the one-point (at "∞") compactification of $\mathscr{S}$ and consider the space $C(\bar{\mathscr{S}})$. Now $C_0(\mathscr{S})$ can be identified with the subspace $\{f \in C(\bar{\mathscr{S}}) : f(\infty) = 0\}$. Since $\tilde{T} : C_0(\mathscr{S}) \to \mathscr{X}$ is continuous and $C_0(\mathscr{S})$ is an "abstract M-space", there is a continuous operator $\bar{T} : C(\bar{\mathscr{S}}) \to \mathscr{X}$ such that $\bar{T} \mid C_0(\mathscr{S}) = \tilde{T}$. This follows from the fact that for any Banach space $\mathscr{Z}$ containing a subspace which is an abstract M-space, there is a projection of norm one on $\mathscr{Z}$ onto that subspace, by the well-known Kelley-Nachbin-Goodner theorem (cf. e.g., [8], p. 398), and $\bar{T} = \tilde{T} \circ Q$. Hence by the Dunford-Schwartz theorem noted above, there is a vector measure $\tilde{Z}$ on $\bar{\mathscr{S}}$ into $\mathscr{X}$ such that

$$\bar{T}(f) = \int_{\bar{\mathscr{S}}} f(t)\tilde{Z}(dt),\ f \in C(\bar{\mathscr{S}}), \tag{37}$$

and $\| \bar{T} \| = \| \tilde{Z} \|(\bar{\mathscr{S}})$, the integral on the right being in the D-S sense. Define $Z : \mathscr{B}(\mathscr{S}) \to \mathscr{X}$ as $Z(A) = \tilde{Z}(\mathscr{S} \cap A)$, $A \in \mathscr{B}(\mathscr{S})$. Then Z is a vector measure and $\| Z \| \leqslant \| \tilde{Z} \|$. Moreover, if $f_0 = f\,|_{\mathscr{S}}$, then

$$\begin{aligned} \bar{T}(f) &= \int_{\mathscr{S}} f_0(t)Z(dt) + \int_{``\{\infty\}"} f(\infty)\tilde{Z}(dt),\ f \in C(\bar{\mathscr{S}}) \\ &= \tilde{T}(f_0), \qquad \text{since} \quad f(\infty) = 0 . \end{aligned}$$

Hence $\bar{T}(f) = \tilde{T}(f)$, $f \in C_0(\mathscr{S})$ with $\| \tilde{T} \| \leqslant \| \bar{T} \| = \| \tilde{T}Q \| \leqslant \| \tilde{T} \|$, and

$$\tilde{T}(f) = \int_{\mathscr{S}} f(t)Z(dt),\ f \in C_0(\mathscr{S}) . \tag{38}$$

Thus writing $\mathbf{R}$ for $\mathscr{S}$ from now on (the above general case is needed later), it follows that

$$\begin{aligned} \| \tilde{T} \| = \sup \{\| \textstyle\int_{\mathbf{R}} f(t)Z(dt) \| : f \in C_0(\mathbf{R}), \| f \|_u \leqslant 1\} &= \| Z \|(\mathbf{R}) \\ &= \| \tilde{Z} \|(\bar{\mathbf{R}}) , \end{aligned}$$

and T and Z correspond to each other uniquely. Since $\tilde{T} \mid \mathscr{Y} = T$, this implies

$$T(\hat{f}) = \int_{\mathbf{R}} \hat{f}(t)Z(dt) = \int_{\mathbf{R}} f(t)X(t)dt,\ f \in L^1(\mathbf{R}) , \tag{39}$$

and $\| T \| = \| Z \|(\mathbf{R})$.

Let $l \in \mathscr{X}^*$. Then (39) becomes (since a continuous operator commutes with the D-S integral, cf. [8], p. 324 and p. 153, and $\mathscr{X}^*$ is the adjoint space of $\mathscr{X}$),

$$\int_{\mathbf{R}} \hat{f}(t)\, l \circ Z(dt) = \int_{\mathbf{R}} f(t)\, l \circ X(t)dt . \tag{40}$$

In (40) now both are ordinary Lebesgue integrals, and hence using the Fubini theorem (for signed measures) on the left one has:

$$\int_{\mathbf{R}} f(t)dt \int_{\mathbf{R}} e_t(\lambda) l \circ Z(d\lambda) = \int_{\mathbf{R}} f(t) l \circ X(t)dt .$$

Subtracting and using the same theorem of ([8], p. 324),

$$\int_{\mathbf{R}} f(t) l\left(\int_{\mathbf{R}} e_t(\lambda) Z(d\lambda) - X(t)\right)dt = 0, \; l \in \mathcal{X}^*, \; f \in L^1(\mathbf{R}) . \tag{41}$$

It follows that the coefficient of f vanishes *a.e.*, (everywhere as it is continuous). Since $l \in \mathcal{X}^*$ is arbitrary it finally results that the quantity inside l is zero, for each $t \in \mathbf{R}$. Thus

$$X(t) = \int_{\mathbf{R}} e_t(\lambda) Z(d\lambda) = \int_{\mathbf{R}} e^{it\lambda} Z(d\lambda), \; t \in \mathbf{R} . \tag{42}$$

Hence X is weakly harmonizable by Theorem 3.3.

For the converse, let $X : \mathbf{R} \to L_0^2(P)$ be weakly harmonizable. Then X admits a representation of (42) by Theorem 3.3. Since $\| Z \| (\mathbf{R}) < \infty$, (21) implies $\| X(t) \|_2 \leqslant M_0 < \infty$ for all $t \in \mathbf{R}$, and as $l \circ X(\cdot)$ is the Fourier transform of $l \circ Z$, $l \in \mathcal{X}^*$, X is weakly continuous. Consider the Bochner integral for $(fX)(\cdot)$ as

$$l\left(\int_{\mathbf{R}} f(t)X(t)dt\right) = \int_{\mathbf{R}} f(t) l \circ X(t)dt = \int_{\mathbf{R}} f(t) \cdot \int_{\mathbf{R}} e_t(\lambda)(l \circ Z)(d\lambda)dt , \tag{43}$$

since $l \circ X$ is the Fourier transform of a signed measure

$$= \int_{\mathbf{R}} \int_{\mathbf{R}} f(t) e_t(\lambda) l \circ Z(d\lambda)dt, \text{ by Fubini's theorem,}$$

$$= \int_{\mathbf{R}} \hat{f}(\lambda) l \circ Z(d\lambda)$$

$$= l\left(\int_{\mathbf{R}} \hat{f}(\lambda) Z(d\lambda)\right), \text{ by ([8], p. 324) again.} \tag{44}$$

Since $l \in \mathcal{X}^*$ is arbitrary, (44) implies

$$\int_{\mathbf{R}} f(t)X(t)dt = \int_{\mathbf{R}} \hat{f}(\lambda) Z(d\lambda) \in \mathcal{X} . \tag{45}$$

Hence, using (21), one has

$$\left\| \int_{\mathbf{R}} f(t)X(t)dt \right\|_2 \leqslant \| \hat{f} \|_u \| Z \|(\mathbf{R}) = c \| \hat{f} \|_u , \quad f \in L^1(\mathbf{R}) , \tag{46}$$

where $c = \| Z \| (\mathbf{R}) < \infty$. It therefore follows that the set

$$\left\{\int_{\mathbf{R}} f(t)X(t)dt : \| \hat{f} \|_u \leqslant 1, f \in L^1(\mathbf{R})\right\} \subset L_0^2(P) ,$$

and is bounded. Since $\mathcal{X}$ is reflexive, X is V-bounded. This completes the proof.

Remarks. 1. Since V-boundedness concept is defined for general Banach spaces (for a treatment of this case, cf. [33]), and its Hilbert space version is equivalent to weak harmonizability, by the above theorem, *the latter term will be used in the Hilbert space context.* (Using the general definition of V-boundedness, a characterization of a process $X : \mathbf{R} \to \mathscr{X}$, a reflexive space, which is a Fourier transform of a vector measure is given in Theorem 7.2 below. It extends a result of [12].)

2. The preceding proof is arranged so that if $\mathbf{R}$ is replaced by a locally compact abelian (LCA) group G, the result and proof hold with essentially no change. The functions $\{e_t(\cdot), t \in G\}$ will then be group characters. Thus the result takes care of $G = \mathbf{R}^n$; so the (weakly) harmonizable random fields are included. Precise statements and further results in the general case will be given later.

If $\mathscr{W}$ is the set of all weakly harmonizable processes on $\mathbf{R} \to L_0^2(P) = \mathscr{X}$, and $T \in B(\mathscr{X})$, the algebra of bounded linear operators on $\mathscr{X}$, then $Y(t) = TX(t)$, $t \in \mathbf{R}$ defines a process which can be written as:

$$Y(t) = T\left(\int_{\mathbf{R}} e^{it\lambda} Z(d\lambda)\right) = \int_{\mathbf{R}} e^{it\lambda} (T \circ Z)(d\lambda), \tag{47}$$

by ([8], p. 324), and it can be seen that $\tilde{Z} = T \circ Z : \mathscr{B} \to \mathscr{X}$ is a stochastic measure, $\| \tilde{Z} \| (\mathbf{R}) \leqslant \| T \| \| Z \| (\mathbf{R}) < \infty$. Hence $Y \in \mathscr{W}$. Thus one has:

COROLLARY 4.3. *$B(\mathscr{X}) \cdot \mathscr{W} = \mathscr{W}$, or in words, the linear space of weakly harmonizable processes is a module over the class of all bounded linear transformations on $\mathscr{X} = L_0^2(P)$.*

Since each stationary process X is trivially strongly (hence weakly) harmonizable, if $P : \mathscr{X} \to \mathscr{X}$ is any orthogonal projection, then $Y = PX \in \mathscr{W}$, i.e. weakly harmonizable by Corollary 4.3. In particular if $\{X_n, n \in Z\} \subset \mathscr{X}$ is an orthonormal sequence, $\mathscr{X}_0 = \overline{sp}(X_n, n > 0)$, let $Q(\mathscr{X}) = \mathscr{X}_0$ be the orthogonal projection and $Y_n = QX_n = X_n$ if $n > 0$, $= 0$ if $n \leqslant 0$. The process $\{Y_n, n \in \mathbf{Z}\} \in \mathscr{W}$, but it is *not* strongly harmonizable. Thus the class of weakly harmonizable processes is strictly larger than the strongly harmonizable class. (The latter is not a module over $B(\mathscr{X})$.)

In spite of the above comment, each weakly harmonizable process can be approximated "pointwise" by a sequence of strongly harmonizable ones. This observation is essentially due to Niemi [29]. The precise result is as follows:

THEOREM 4.4. *Let $X : \mathbf{R} \to L_0^2(P)$ be a weakly harmonizable process. Then there exists a sequence of strongly harmonizable processes $X_n : \mathbf{R} \to L_0^2(P)$ such that $X_n(t) \to X(t)$, as $n \to \infty$, in $L_0^2(P)$ uniformly (in t) on compact subsets of $\mathbf{R}$. If $\mathbf{R}$ is replaced by an LCA group G the same result holds with $\{X_n, n \in I\}$ being a net of such process. (The convergence is here in $L^2(P)$-mean.)*

Proof. By hypothesis, there is a stochastic measure $Z : \mathscr{B} \to \mathscr{X} = L_0^2(P)$, such that

$$X(t) = \int_{\mathbf{R}} e_t(\lambda) Z(d\lambda), \qquad t \in \mathbf{R}.$$

Thus $X : \mathbf{R} \to \mathscr{X}$ is a continuous mapping. If $\mathscr{H}_X = \overline{sp}\{X(t), t \in \mathbf{R}\} \subset \mathscr{X}$, then the continuity of X (and the separability of $\mathbf{R}$) implies $\mathscr{H}_X$ is separable. Hence there exists a sequence $\{\varphi_n, n \geqslant 1\} \subset \mathscr{X}_X$ which is a complete orthonormal (CON) basis for $\mathscr{X}_X$, so that

$$X(t) = \sum_{n=1}^{\infty} \varphi_n (X(t), \varphi_n), \qquad t \in \mathbf{R}, \tag{48}$$

the series converging in the (norm) topology of $\mathscr{H}_X$ for each t. Define

$$X_n(t) = \sum_{k=1}^{n} \varphi_k (X(t), \varphi_k), \qquad t \in \mathbf{R}. \tag{49}$$

Claim: $\{X_n(t), t \in \mathbf{R}\}$, $n \geqslant 1$, is the desired sequence. [In the general LCA group case $\{\varphi_n, n \in I\}$ is a net of CON elements of $\mathscr{H}_X$, since G, hence $\mathscr{H}_X$, need not be separable. Otherwise the same argument works with trivial modifications.]

To verify the claim, it is clear that $X_n(t) \to X(t)$ in $\mathscr{H}_X$ for each $t \in \mathbf{R}$. To see that X_n is strongly harmonizable, let

$$l_k : X \mapsto (X, \varphi_k), \qquad X \in \mathscr{H}_X.$$

Then $l_k \in \mathscr{H}_X^*$ for each k. Hence using the standard properties of the D-S integral, one has

$$X_n(t) = \sum_{k=1}^{n} \varphi_k l_k(X(t)) = \sum_{k=1}^{n} \varphi_k \cdot l_k\left(\textstyle\int_{\mathbf{R}} e_t(\lambda) Z(d\lambda)\right),$$

since X is weakly harmonizable,

$$= \sum_{k=1}^{n} \varphi_k \textstyle\int_{\mathbf{R}} e_t(\lambda) l_k \circ Z(d\lambda) = \int_{\mathbf{R}} e_t(\lambda) \zeta_n(d\lambda), \tag{50}$$

where $\zeta_n(\cdot) = \sum_{k=1}^{n} \varphi_k l_k \circ Z(\cdot)$. Let $G_n(A, B) = (\zeta_n(A), \zeta_n(B))$. Then G_n is of finite total variation. Indeed, if $\mu_k = l_k \circ Z$, which is a signed measure (hence has finite variation) on $\mathbf{R}$, let

$$\eta_k(A, B) = (\varphi_k \mu_k(A), \varphi_k \mu_k(B)) = \mu_k(A)\overline{\mu_k(B)}.$$

So $G_n(A, B) = \sum_{k=1}^{n} \mu_k(A)\overline{\mu_k(B)}$. Since

$$|\mu_k(A)|\,|\mu_k(B)| \leqslant (|\mu_k|(\mathbf{R}))^2 < \infty$$

for each k, it follows that each η_k and hence G_n for each n has finite variation so that each X_n is strongly harmonizable.

It was already noted that X being weakly harmonizable, it is strongly continuous. [This is true even if $\mathbf{R}$ is replaced by an LCA group G (cf. [21], p. 270).] So if $K \subset \mathbf{R}$ is a compact set, then its image $X(K) \subset \mathscr{H}_X \subset L_0^2(P)$ is also (norm) compact. But $\mathscr{H}_X$ being a Hilbert space it has the (metric) approximation property. [This means the identity on $\mathscr{H}_X$ can be uniformly approximated by a sequence (net) of (contractive) degenerate, or finite rank, operators on each compact subset of $\mathscr{H}_X$.] Then $X_n(t) \to X(t)$ in $\mathscr{X}$ for each $t \in \mathbf{R}$ implies, by a result in Abstract Analysis in the presence of the approximation property, that the convergence holds in $\mathscr{X}$ uniformly on compact subsets of $\mathscr{X}$. This and the fact that $X(K)$ is compact implies that $X_n(t) \to X(t)$ in $L_0^2(P)$, uniformly for $t \in K \subset \mathbf{R}$. In the general LCA case, the same holds with nets replacing sequences. This completes the proof.

Remark. Even though the weakly harmonizable process is bounded and weakly (hence strongly here) continuous with some nice closure properties demonstrated above, it does not exhaust the class of all bounded continuous processes in $L_0^2(P)$. This can be seen from Theorem 3.2 by a suitable choice of a vector measure of finite local semivariation but which is not of finite semivariation. The following example demonstrates this point. Let $L^1(\mathbf{R})$ be identified with $\mathscr{M}(\mathbf{R})$ of regular signed measures on $\mathbf{R}$ by the Radon-Nikodým theorem (i.e. $f \in L^1(\mathbf{R}) \leftrightarrow \int_{(\cdot)} f(t)dt \in \mathscr{M}(R)$). Now it is known that there are nontrivial functions in $C_0(\mathbf{R}) - \mathscr{Y}_1$ where $\mathscr{Y}_1 = \{\hat{\mu} : \mu \in \mathscr{M}(\mathbf{R})\}$. Let $f \in C_0(\mathbf{R}) - \mathscr{Y}$. For instance

$$f(x) = \operatorname{sgn}(x)\left((\log|x|)^{-1}\chi_{[|x| \geq e]} + \frac{|x|}{e}\chi_{[|x|<e]}\right), \qquad x \in \mathbf{R},$$

is known to be such an f. Let $\varphi \in L_0^2(P)$, $\|\varphi\|_2 = 1$. Let $l \in (L_0^2(P))^*$ such that $l(\varphi) = 1$. Consider the trivial process $X_0 : t \mapsto f(t)\varphi$. Then $X_0 : \mathbf{R} \to L_0^2(P)$ is bounded and continuous but not weakly harmonizable, since otherwise there exists a stochastic measure Z such that (by Theorem 3.3)

$$X_0(t) = \int_{\mathbf{R}} e_t(\lambda) Z(d\lambda), \qquad \text{and}$$

$$f(t) = l(X_0(t)) = \int_{\mathbf{R}} e_t(\lambda)\,(l \circ Z)\,(d\lambda).$$

Since $l \circ Z \in \mathscr{M}(\mathbf{R})$, this would contradict the choice of f.

Here is an interesting consequence of the preceding theorem.

THEOREM 4.5. *Let* $X : \mathbf{R} \to L_0^2(P)$ *be a weakly harmonizable process and let* $Z : \mathscr{B} \to L_0^2(P)$ *be its representing measure by (30). Then there is (nonuniquely) a fixed sequence of finite regular Borel measures* $\beta_n : \mathscr{B} \to \mathbf{R}^+$ *such that for each* $f \in C_0(\mathbf{R})$,

$$\| \int_{\mathbf{R}} f(t) Z(dt) \|_2 \leqslant \liminf_n \| f \|_{2, \beta_n}$$

$$\left(= \liminf_n \left[\int_{\mathbf{R}} | f(t) |^2 \beta_n(dt)\right]^{1/2}\right). \tag{51}$$

Remark. Even though this result is deducible from the general Theorem 5.5 below, the present proof is elementary and has some interest and will be given here. It leads to the general case.

Proof: By hypothesis, $X(\cdot)$ is represented by a stochastic measure Z (cf. (30)), and by the preceding theorem there are strongly harmonizable $X_n \to X$, uniformly on compact subsets of $\mathbf{R}$. Let ζ_n be the representing measure of X_n, so that $\zeta_n, Z : \mathscr{B} \to L_0^2(P)$, and

$$\int_{\mathbf{R}} f(\lambda) Z(d\lambda) = \lim_{n \to \infty} \int_{\mathbf{R}} f(\lambda) \zeta_n(d\lambda), \tag{52}$$

the limit existing in $L_0^2(P)$ when f is a trigonometric polynomial. Since such polynomials separate points of $\mathbf{R}$ and so are uniformly dense in $C_0(\mathbf{R})$, and the integrals in (52) define bounded operators from $C_0(\mathbf{R})$ into $L_0^2(P)$, it follows that (52) holds for all $f \in C_0(\mathbf{R})$, by standard reasoning (cf. [8], II.3.6). Hence

$$\begin{aligned} \alpha_0^f = \| \textstyle\int_{\mathbf{R}} f(\lambda) Z(d\lambda) \|_2^2 &= \lim_{n \to \infty} \| \textstyle\int_{\mathbf{R}} f(\lambda) \zeta_n(d\lambda) \|_2^2, \quad f \in C_0(\mathbf{R}) \\ &= \lim_{n \to \infty} \textstyle\int_{\mathbf{R}} \int_{\mathbf{R}} f(\lambda) \overline{f}(\lambda') F_n(d\lambda, d\lambda'), \end{aligned} \tag{53}$$

where $F_n(s, t) = \big(\zeta_n(-\infty, s), \zeta_n(-\infty, t)\big)$ is a covariance function of bounded variation for each n. Let $| F_n | (\cdot, \cdot)$ be the (Vitali) variation measure of the bimeasure F_n. Then the hermitian property of F_n implies, in an obvious notation, $| F_n | (A, B) = | F_n | (B, A)$. Now define a mapping $\beta_n : \mathscr{B} \to \mathbf{R}^+$ by the equation:

$$\beta_n(A) = | F_n | (A, \mathbf{R}) = \frac{1}{2} \{ | F_n | (A, \mathbf{R}) + | F_n | (\mathbf{R}, A) \}, \qquad A \in \mathscr{B},$$

so that β_n is a finite Borel measure, and

$$\int_{\mathbf{R}} f(\lambda) \beta_n(d\lambda) = \frac{1}{2} \left[\int_{\mathbf{R}} \int_{\mathbf{R}} f(s) \; | F_n | (ds, dt) + \int_{\mathbf{R}} \int_{\mathbf{R}} f(t) \, | F_n | (ds, dt)\right]. \tag{54}$$

Since F_n is positive (semi-) definite,

$$0 \leqslant \int_{\mathbf{R}} \int_{\mathbf{R}} f(s)\overline{f(t)} F_n(ds, dt) \leqslant \int_{\mathbf{R}} \int_{\mathbf{R}} |f(s)\overline{f(t)}| \, |F_n|(ds, dt)$$

$$\leqslant \frac{1}{2} [\int_{\mathbf{R}} \int_{\mathbf{R}} |f(s)|^2 \, |F_n|(ds, dt) + \int_{\mathbf{R}} \int_{\mathbf{R}} |f(t)|^2 \, |F_n|(ds, dt)],$$

$$\text{since} \quad |ab| \leqslant (|a|^2 + |b|^2)/2,$$

$$= \int_{\mathbf{R}} |f(s)|^2 \beta_n(ds), \qquad \text{by (54)}. \tag{55}$$

This and (53) yield

$$\alpha_0^f = \| \int_{\mathbf{R}} f(\lambda) Z(d\lambda) \|_2^2 = \lim_n \int_{\mathbf{R}} \int_{\mathbf{R}} f(\lambda)\overline{f}(\lambda') F_n(d\lambda, d\lambda')$$

$$\leqslant \liminf_n \int_{\mathbf{R}} |f(\lambda)|^2 \beta_n(d\lambda), \qquad f \in C_0(\mathbf{R}). \tag{56}$$

This completes the proof.

Remark. For a deeper analysis of the structure of these processes, it is desirable to replace the sequence $\{\beta_n, n \geqslant 1\}$ by a single Borel measure. This is nontrivial. In the next section for a more general version, including harmonizable fields, such a result will be obtained.

5. Domination problem for harmonizable fields

The work of the preceding section indicates that the weakly harmonizable processes are included in the class of functions which are Fourier transformations of vector measures into Banach spaces. A characterization of such functions, based on the V-boundedness concept of [2], has been obtained first in [33]. For probabilistic applications (e.g., filtering theory) the domination problem, generalizing Theorem 4.5, should be solved. The following result illuminates the nature of the general problem under consideration.

Theorem 5.1. *Let (Ω, Σ) be a measurable space, $\mathcal{X}$ a Banach space and $\nu : \Sigma \to \mathcal{X}$ be a vector measure. Then there exists a (finite) measure $\mu : \Sigma \to \mathbf{R}^+$, a continuous convex function $\varphi : \mathbf{R}^+ \to \mathbf{R}^+$ such that $\frac{\varphi(x)}{x} \nearrow \infty$ as $x \nearrow \infty$, $\varphi(0) = 0$, and ν has φ-semivariation finite relative to μ in the sense that*

$$\| \nu \|_{\varphi}(\Omega) = \sup \{ \| \int_{\Omega} f(\omega) \nu(d\omega) \|_{\mathcal{X}} : \| f \|_{\psi, \mu} \leqslant 1 \} < \infty, \tag{57}$$

where $\| f \|_{\psi, \mu} = \inf \{ \alpha > 0 : \int_{\Omega} \psi \left(\frac{|f(\omega)|}{\alpha} \right) \mu(d\omega) \leqslant 1 \} < \infty,$ *and the*