Introduction

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LOCALLY HOMOGENEOUS VARIATIONS OF HODGE STRUCTURE

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INTRODUCTION

It was my original goal, in working on this paper, to show the link between the Hodge theory of vector-valued forms on (compact) quotients of Hermitian symmetric spaces to the Hodge theory for local systems that underlie polarized variations of Hodge structure. The first was the object of study over fifteen years ago, notably in the work of Matsushima, Murakami and Kuga; the second is an unpublished construction of Deligne that has been described and used in my recent work [11]. This paper still fulfills its expository function, and it should serve to unify related ideas, juxtaposing techniques from representation theory and transcendental algebraic geometry. However, it now seems likely that each subject will benefit from techniques and ideas drawn from the other, as attested by the results in Sections 4 and 5. A starting point for this overlap already appears in [11, §12].

We begin the description of the subject matter of this paper. Let $\mathbf{V}$ be a locally constant sheaf on the compact Kähler manifold S, with $\mathbf{V}$ underlying a polarized variation of Hodge structure. One can decompose the $\mathbf{V}$-valued forms on S into components of type $(p, q); (r, s)$: (p, q)-forms with values in the (r, s) Hodge decomposition bundle. [Note that $r + s$ will be equal to the weight of the variation of Hodge structure, and $p + q$ will usually be held fixed (though arbitrary), so there are really only two independent parameters.] Then, according to Deligne, the harmonic forms, and therefore the cohomology $H^{\bullet}(S, \mathbf{V})$, decompose according to "total" bidegree (P, Q), where $P = p + r$ and $Q = q + s$.

Now let G be a real semi-simple Lie group with finite center, K a maximal compact subgroup, Γ a cocompact discrete subgroup of G, $S = \Gamma\backslash G/K$, and

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(ρ, V) a finite-dimensional representation of G. There is a simple construction of a locally constant sheaf $\mathbf{V}$ on S, such that there is a natural isomorphism

$$H^\bullet(\Gamma, V) \simeq H^\bullet(S, \mathbf{V})$$

If G/K is *Hermitian* symmetric, then according to Matsushima and Murakami [7] the harmonic forms decompose according to (p, q) type. It was this apparent variance with Deligne's result that aroused my interest in this material.

There is a natural "locally homogeneous" (complex) variation of Hodge structure (4.9) on S, determined by the decomposition of V into character spaces under the center of K. The variation is *real* if (ρ, V) is. By combining Deligne's theory and that of [7], we see that there is a complete decomposition of harmonic forms into (p, q); (r, s) components in this case. It is also possible to see this directly from the identities (3.12) among the various Laplacians. In fact, an even finer decomposition is possible (see (3.29)), similar to the one in the main theorem of [2]. We draw algebraic consequences in hypercohomology in (5.16) and (5.23).

Ultimately, one would like to understand the cohomology in the case of *non-compact* locally symmetric varieties S (of finite volume). In most cases, according to [18], this is tantamount to saying that Γ is an arithmetic subgroup of G. The Hodge-theoretic techniques are, in a sense, "formal", and they yield a decomposition theorem for the intrinsic L_2 cohomology $H^\bullet_{(2)}(S, \mathbf{V})$ (where exactness conditions must involve only L_2 forms) relative to natural metrics. The precise relation between the L_2 and total cohomology groups is only beginning to emerge (see [12]). The case $G = SL(2, \mathbf{R})$, where S is an algebraic curve, has been treated in [11]. Here, the L_2 cohomology is naturally isomorphic to some "topological" cohomology $H^\bullet(\bar{S}, \bar{\mathbf{V}})$ on the smooth compactification $\bar{S}$ of S, where $\bar{\mathbf{V}}$ is a certain extension to $\bar{S}$ of the sheaf $\mathbf{V}$ (see [11, §6]). I suspect that it is possible to describe the L_2 cohomology for arbitrary G in terms of data on a suitable compactification of S (see [12, (3.99)]).

The use of Deligne's Hodge decomposition in the locally homogeneous case permitted us in [11, §12] to arrive at an explanation of the "mysterious" isomorphism of (Eichler-) Shimura. Let $\mathbf{V}$ be $\mathrm{Symm}^k\,\mathbf{C}^2$, as a representation of $SL(2, \mathbf{R})$. The parabolic cohomology $H^1_P(\Gamma, V)$ is naturally isomorphic to $H^1(\bar{S}, \bar{\mathbf{V}})$, and the cusp forms of weight $k + 2$ for Γ determine $\mathbf{V}$-valued holomorphic 1-forms on S that are L_2 in the Poincaré (Bergman) metric. This gives the $(k+1, 0)$-component *(sic)* of the Hodge structure, and one can see directly that its complex conjugate is the only other term which can be non-zero.

In this paper, we prove for arbitrary G the cohomological result (5.29) which underlies the Shimura isomorphism in the case of $SL(2, \mathbf{R})$. It is most easily stated in terms of the holomorphic de Rham complex $\Omega^\bullet_S(\mathbf{V})$.The Deligne Hodge filtration (associated to the (P, Q) decomposition) is placed on $\Omega_S^\bullet(\mathbf{V})$, and we are able to determine the cohomology sheaves for the successive quotients as [the locally-free sheaves of sections of] locally homogeneous vector bundles associated to certain representations of K. Of course, $\Omega_S^\bullet(\mathbf{V})$ is itself comprised of homogeneous vector bundles, and the main point is to recognize that each successive quotient corresponds to a single and distinct character under the action of a certain subgroup of the center of K. The representations of K which occur in the cohomology sheaves are none other than the character spaces in the Lie algebra cohomology $H^\bullet(\mathfrak{p}^+, V)$. From this, we are also able to illustrate how, for real representations, cohomology vanishing theorems might be proved by exploiting the fundamental role of the center of K (see (5.34) ff.). However, this method does not seem to produce new results.

From the point of view of the transcendental algebraic geometer, locally homogeneous variations of Hodge structure are interesting as a class of examples of variations of Hodge structure in several variables. In the case of one variable, Schmid's SL_2-orbit theorem [19] can be interpreted as saying that every variation of Hodge structure is asymptotic, near a singularity, to a locally homogeneous one for $SL(2, \mathbf{R})$. This gives a fairly complete description in one variable. However, there is currently no generalization to several variables, nor are the asymptotics of the Hodge norm (a corollary of the SL_2-orbit theorem in the one-variable case) understood. Since the locally homogeneous variations of Hodge structure are so explicit, it should be possible to calculate the asymptotics directly, relative to a nice compactification of S. Hopefully, this will give us a clue to the general situation; it will certainly provide a lower bound for the content of a general theory.

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