

4. Generalized algorithm for power-group algebras

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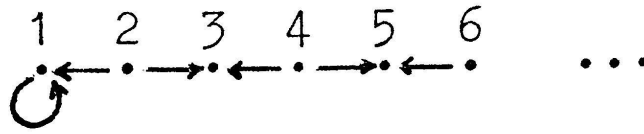
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$Z_{XXXX} \vee XXXX \notin M_{\{J, K\}}$ and so $M_{\{J, K\}}$ is infinite, since for the infinite graph shown below:



$(JK)^m$ adds all edges $\langle i, j \rangle$ with $i, j \leq 2m$

and $(JK)^m J$ adds all edges $\langle i, j \rangle$ with $i, j \leq 2m + 1$

Therefore $J, JK, JKJ, \dots$ are all distinct.

4. GENERALIZED ALGORITHM FOR POWER-GROUP ALGEBRAS

To elucidate the correctness of the algorithm and to encompass some more general applications we need to generalize from the $\{0, 1\}$ Boolean algebra to a slightly richer structure. The *power-group algebra* $P(G)$ is a structure defined from an arbitrary group G . The elements of $P(G)$ are the subsets of G ; the operations we require are *union* ($\cup$), *complex product*:

$$ab = \{ gh \mid g \in a, h \in b \} \quad \text{for } a, b \subseteq G$$

and *converse*:

$$\bar{a} = \{ g^{-1} \mid g \in a \}$$

$P(G)$ is a monoid with respect to product with identity $\lambda = \{\text{identity}_G\}$. As before we shall be considering matrices over the structure, with matrix product and union defined in the obvious way from product and union in $P(G)$, and matrix *converse* defined by

$$(\bar{A})_{ij} = \bar{A}_{ji}$$

The key properties of power-group algebras which are needed are given below

LEMMA. Let a, b be elements and A, B matrices

- (i) $\bar{\bar{a}} = a; \bar{\bar{A}} = A$
- (ii) $\overline{ab} = \bar{b}\bar{a}; \overline{AB} = \bar{B}\bar{A}$
- (iii) if $a \neq \emptyset$ then $a\bar{a} \supseteq \lambda; A\bar{A}A \supseteq A$

Proof. We prove only (iii). The first part is immediate and has the consequence that $\bar{a}a \supseteq a$ for all a . For the second part

$$(A\bar{A}A)_{ij} \supseteq A_{ij} \bar{A}_{ji} A_{ij} = A_{ij} \bar{A}_{ij} A_{ij} \supseteq A_{ij}. \quad \square$$

We observe that the $\{0, 1\}$ Boolean algebra is the power-group algebra corresponding to the trivial one-element group. Other groups we shall use are $\langle \mathbf{Z}_k, + \rangle$ and $\langle \mathbf{R}, + \rangle$, the integers modulo k and the reals.

The operators $*$, $+$, V and W are defined just as before for matrices and elements. In the Boolean case we had the trivial results

$$\bar{a} = a^+ = a^W = a$$

and

$$a^* = a^V = 1$$

In the general case we must augment the algorithm a little. Suppose for example there are edges labelled a, b from i to j and k respectively, and a self-loop at i labelled c . Then the label of the edge from j to k must eventually receive a term corresponding to the indirect paths from j to k i.e.

$$\bar{a}(a\bar{a} \vee c \vee \bar{c} \vee b\bar{b})^* b$$

The generalized form of ψ_i is:

$$\begin{aligned} A_{i*} &:= A_{ii}^V A_{i*} \\ k &:= i + 1 \text{ step } 1 \text{ until } (k=n \text{ or } A_{ik} \neq \emptyset) \\ \text{if } A_{ik} \neq \emptyset &\text{ then } A_{k*} := A_{k*} \cup \bar{A}_{ik} A_{i*} \end{aligned}$$

The programs for ψ'_i, Ψ and Ψ' are analogous. They simplify to the programs of section 2 in the Boolean case.

The question of generalization could have been tackled axiomatically. Suppose $i < j < k$ and $A_{ij}, A_{ik} \neq \emptyset$. If row i were sent directly to row k we would have $\bar{A}_{ik} A_{i*}$, whereas via j we get

$$\overline{A_{ij} A_{ik}} \cdot \bar{A}_{ij} A_{i*} = \bar{A}_{ik} A_{ij} \bar{A}_{ij} A_{i*}$$

and we seem to require $A_{ij} \bar{A}_{ij} \geq \lambda$ for correctness. A power-group algebra seems the only structure of possible interest with this property.

Suppose that A is a matrix over $P(G)$ and we compute $A^{RQ} = A^V$. A^V_{ij} is the set of elements of G which are the product of labels from a (weak) path from i to j in the graph corresponding to A . Each diagonal element

A^V_{ii} is a subgroup of G and two diagonal elements in the same (weak) component are conjugate since we may show:

$$A^V_{ii} \cdot g = g \cdot A^V_{jj} = A^V_{ij} \quad \text{for any } g \in A^V_{ij}$$

In particular if all circuits on non-empty edges correspond to the group identity then in A^V each entry has at most one element. For example given a matrix A over $\langle \mathbf{R}, + \rangle$ we determine from A^V for each component of the graph whether there are (weak) circuits with non-zero sums. If all circuits sum to zero then the graph is a "potential" graph, i.e. there is a function $\text{pot} : \mathbf{R} \rightarrow \{\text{vertices}\}$ such that each edge $\langle u, v \rangle$ has the value $(\text{pot}(v) - \text{pot}(u))$. Similarly over $\langle \mathbf{Z}_k, + \rangle$ we may determine whether each weak circuit of a directed graph has zero sum where forward and backward edges are accounted $+1$ and -1 respectively. Naturally we may find it convenient in some cases to hold only a homomorphic image of $P(G)$ for the computations e.g.

$$h(\emptyset) = \emptyset$$

$$h(\{g\}) = g$$

$$h(a) = \omega \quad \text{when} \quad |a| > 1.$$

5. PROOFS OF CORRECTNESS

An operator ϕ is *monotonic* if $A \subseteq B$ implies $A^\phi \subseteq B^\phi$. ψ_i is *not* a monotonic operator but rather surprisingly Ψ is. To simplify our proofs we introduce several monotonic operators. Define ϕ_i by the program

$$A_{i*} := A_{ii}^V A_{i*}$$

$$\text{for } k := i + 1 \text{ step } 1 \text{ until } n \text{ do } A_{k*} := A_{k*} \cup \overline{A_{ik}} A_{i*}$$

and ϕ'_i analogously using " $i - 1$ step -1 until 1 ". Both are obviously monotonic. Φ and Φ' are defined from ϕ_i and ϕ'_i in a similar way to Ψ and Ψ' . Although $\Psi \subseteq \Phi$ is evident, the following result is not.

THEOREM 5. $\Psi = \Phi$ (and $\Psi' = \Phi'$).

Proof. Consider ψ_i applied to an arbitrary matrix A and suppose it selects the index k with $A_{ik} \neq \emptyset$. (If no index is selected then of course $A^{\psi_i} = A^{\phi_i}$). We verify that: