In the first sum we use the bound $| G_T(x/i) | \leqslant 3T/2$, which is easily deduced from (4); for the second sum we calculate $G_T(y)$ directly for $T - 1 \leqslant y < TK$, and let $U = U(T, K)$ and $L = L(T, K)$ denote the upper and lower bounds for G_T on this interval.

Combining the estimates we obtain

$$(5) \quad \psi(x) - \psi\left(\frac{x}{T-1}\right) - \frac{3}{2}T\psi\left(\frac{x}{TK}\right) + L\left\{\psi\left(\frac{x}{T-1}\right) - \psi\left(\frac{x}{TK}\right)\right\} \leqslant x m_1(T) + O(T \log ex),$$

$$(6) \quad \psi(x) - \psi\left(\frac{x}{T-1}\right) + \frac{3}{2}T\psi\left(\frac{x}{TK}\right) + U\left\{\psi\left(\frac{x}{T-1}\right) - \psi\left(\frac{x}{TK}\right)\right\} \geqslant x m_1(T) + O(T \log ex).$$

We give an upper estimate of $\psi(x)/x$ using (5) with $T = 100$, $TK = 8911$, $L \geqslant -4.9054$, $m_1(100) \leqslant 1.00104$, and Chebyshev's bound $\limsup \psi(x)/x < 1.1056$. We find that $\limsup \psi(x)/x < 1.085$. We estimate $\psi(x)/x$ from below by using (6) with $T = 101$, $TK = 17749$, $U \leqslant 7.2930$, $m_1(101) \geqslant 1.00134$ and the preceding upper estimate of $\psi(x)/x$. We find that $\liminf \psi(x)/x > .924$.

Might Chebyshev have improved his bounds for $\psi(x)/x$ if he had used this method? We must report that that is quite unlikely, because considerable calculation was needed to obtain the modest improvement we have achieved.

NOTE ADDED IN PROOF. Diamond and Kevin Mc Curley have found another sharp elementary estimation method. Their article "Constructive elementary estimates for $M(x)$" will appear in *Number Theory — Proceedings of a conference held at Temple University, May 1980*, Lectures Notes in Math., Springer-Verlag, Berlin.

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