7. Volume

PROPOSITION 5.1 (bis). *The following statement is equivalent to those listed above.*

(d) *The local fundamental group of* V *is finite.*

It is shown in [Prill, p. 381; Brieskorn 2, p. 344] that conditions (a) and (d) are equivalent.

Characterization A6. The local fundamental group of $f^{-1}(0)$ is finite.

Thus Characterizations A5 and A6 are equivalent.

There is an algorithm for computing the local fundamental group of V from a resolution [Mumford], and singularities V with finite, nilpotent and solvable local fundamental group have been classified [Brieskorn 2; Wagreich 2]. When V is a complete intersection, this classification is particularly simple [Durfee 2, Proposition 3.3].

7. VOLUME

Let $f(x, y, z)$ be the germ at the origin $\mathbf{0}$ of a complex analytic function, and suppose that $f(\mathbf{0}) = 0$ and that the origin is an isolated critical point of f. There is an $\varepsilon > 0$ such that $f^{-1}(\mathbf{0})$ intersects all spheres of radius ε' about $\mathbf{0}$ transversally for $0 < \varepsilon' \leqslant \varepsilon$. (See Section 12.) For $t \in \mathbf{C}$, let

$$V_t = f^{-1}(t) \cap D_\varepsilon^6$$

where D_ε^6 is the closed disk of radius ε about $\mathbf{0}$. The function $f(x, y, z)$ takes the constant value t on V_t, so $\frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz \equiv 0$ there. Hence a nowhere-vanishing holomorphic two-form ω_t on V_t may be defined by the equivalent expressions

$$\omega_t = \frac{dy \wedge dz}{\partial f / \partial x} = \frac{dz \wedge dx}{\partial f / \partial y} = \frac{dx \wedge dy}{\partial f / \partial z},$$

Characterization A7. The integral $\int_{V_0} \omega_0 \wedge \bar{\omega}_0$ is finite.

Note that the form $\omega_0 \wedge \bar{\omega}_0$ takes positive real values. The equivalence of Characterizations A2 and A7 is due to Laufer, and follows easily from his expression for the geometric genus in terms of forms [Laufer 2, Corollary 3.6].

A different formulation of this characterization is due to E. Looijenga (unpublished): Let $\Delta(r) = \{t \in \mathbf{C} : t < r\}$, let

$$X(r) = f^{-1}(\Delta(r)) \cap D_\varepsilon^6$$

and let $\operatorname{vol}(X(r))$ be its volume in $\mathbf{C}^3$.

Characterization A7′. $\lim_{r\to 0} r^{-2} \operatorname{vol}(X(r))$ is finite.

Let $\omega = dx \wedge dy \wedge dz$, and note that $\omega \wedge \bar{\omega}$ is $8/i$ times the volume form of $\mathbf{C}^3$. Characterizations A7 and A7′ are equivalent since

$$\lim_{r\to 0} \frac{1}{r^2} \operatorname{vol}(X(r)) = \lim_{r\to 0} \frac{i}{8r^2} \int_{X_r} \omega \wedge \bar{\omega}$$

$$= \lim_{r\to 0} \frac{i}{8r^2} \int_{\Delta(r)} \left(\int_{V_t} \omega_t \wedge \bar{\omega}_t \right) dt \wedge \overline{dt},$$

but since

$$\int_{\Delta(r)} \left(\frac{i}{2}\right) dt \wedge \overline{dt} = \operatorname{vol}(\Delta(r)) = 2\pi r^2 ,$$

the above limit equals

$$\frac{\pi}{2} \int_{V_0} \omega_0 \wedge \bar{\omega}_0 .$$

B. Nine characterizations of simple critical points

We switch our attention from the analytic set defined by the zero locus of an analytic function $f(x, y, z)$ to the function itself and the nature of its critical point. We also generalize to functions $f(z_0, \ldots, z_n)$ of an arbitrary number of variables. The characterizations in the following theorem will start in Section 9.

Theorem B. *Let $f(z_0, \ldots, z_n)$ with $n \geqslant 1$ be the germ at the origin $\mathbf{0}$ of a complex analytic function, and suppose further that $f(\mathbf{0}) = 0$ and that $\mathbf{0}$ is an isolated critical point of f. Then Characterizations B1 through B9 are equivalent.*

8. The classification of right equivalence classes

Let $\mathcal{O}$ be the set of germs f at the origin $\mathbf{0}$ of complex analytic functions on $\mathbf{C}^{n+1}$. (In other words, $\mathcal{O}$ is just the ring $\mathbf{C}\{z_0, \ldots, z_n\}$ of convergent power series.) The ring $\mathcal{O}$ is local with maximal ideal

$$\mathfrak{m} = \{f \in \mathcal{O} : f(\mathbf{0}) = 0\}.$$