

3. CO-DIRECTIONS IN PROJECTIVE SPACE

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is one-to-one on an open neighborhood U of x_0 in G/P and $(\exp X) \cdot x_0$ is identifiable as the point (x_1, \dots, x_n) and the incident hyperplane

$$x'_n - x_n = p_1(x'_1 - x_1) + \dots + p_{n-1}(x'_{n-1} - x_{n-1}).$$

Now, $(\exp X) \cdot x_0 \rightarrow (\exp X) \cdot b_0$ is a section of the bundle G/P_1 over U and, via this section, the form ω on G/P_1 pulls down to

$$\omega_0((\exp X)^{-1} d(\exp X))$$

which, when expressed in terms of $x_1, \dots, x_n, p_1, \dots, p_{n-1}$, will be identified with

$$dx_n - p_1 dx_1 - \dots - p_{n-1} dx_{n-1}$$

up to a constant multiple $a \neq 0$. For this latter calculation we will use

$$\begin{aligned} (\exp X)^{-1} d(\exp X) &= \frac{1 - e^{-ad X}}{ad X} (dX) \\ &= dX - \frac{1}{2} [X, dX] + \frac{1}{6} [X, [X, dX]] - \dots \end{aligned}$$

[4, (10.2)], a series which is finite since \mathfrak{m} is nilpotent. In fact, our choice of X will make the series for $\exp X$ themselves finite. The constant $a \neq 0$ could be made unity by using instead the section $(\exp X) \cdot x_0 \rightarrow (\exp X)g^{-1} \cdot b_0$, where g in P is chosen so that $\chi(g) = a$. This amounts to following the original section by R_a^{-1} in the bundle.

3. CO-DIRECTIONS IN PROJECTIVE SPACE

The contact structure on the $(2n-1)$ -dimensional space of co-directions in complex projective space P^n , described in 2.5, is obtained when the construction of 2.10 is carried out for the simple complex Lie algebra of type $A_n, n \geq 1$.

3.1 Let $\mathfrak{g} = \mathfrak{sl}(n+1; \mathbf{C})$, complex $(n+1)$ by $(n+1)$ matrices of trace zero. For Cartan subalgebra \mathfrak{h} of \mathfrak{g} take the diagonal matrices of \mathfrak{g} . Let $\delta_i, i = 0, 1, \dots, n$ be the linear function on \mathfrak{h} which assigns to $H = \text{diag}(h_1, \dots, h_n)$ in \mathfrak{h} the i^{th} diagonal element: $\delta_i(H) = h_i$. The roots of \mathfrak{g} with respect to \mathfrak{h} are

$$\begin{aligned} \delta_i - \delta_j \quad i, j = 0, 1, \dots, n \\ \text{and } i \neq j \end{aligned}$$

and the root vector E_α corresponding to the root α is

$$E_{\delta_i - \delta_j} = E_{ij},$$

the matrix with 1 in the i^{th} row and j^{th} column and 0s elsewhere [4, (16.2)]. A system of simple roots is

$$\delta_0 - \delta_1, \delta_1 - \delta_2, \dots, \delta_{n-1} - \delta_n,$$

for which the maximal root is

$$\rho = (\delta_0 - \delta_1) + (\delta_1 - \delta_2) + \dots + (\delta_{n-1} - \delta_n) = \delta_0 - \delta_n$$

[4, App., Table E]. The Killing form of \mathfrak{g} is $\langle X, Y \rangle = 2(n+1) \text{tr}(XY)$, but we replace this with $\langle X, Y \rangle = \text{tr}(XY)$ for convenience. Then the H_α in \mathfrak{h} are given by

$$H_{\delta_i - \delta_j} = \text{diag}(0, \dots, 0, 1, 0, \dots, 0, -1, 0, \dots, 0)$$

with 1 and -1 in the i^{th} and j^{th} entry, respectively. Especially,

$$H_\rho = \text{diag}(1, 0, \dots, 0, -1).$$

We have

$$\langle H_\rho, H_{\delta_i - \delta_j} \rangle \begin{cases} < 0 & j = 0 \text{ or } i = n \\ \geq 0 & \text{otherwise} \end{cases},$$

so that \mathfrak{p} in (i) of 2.9 consists of matrices of the form

$$\begin{bmatrix} * & * & \cdots & * \\ & \vdots & & \vdots \\ 0 & & & \\ & * & \cdots & * \\ & & & \\ & & & \\ 0 & \cdots & 0 & * \end{bmatrix}$$

of trace zero, where the starred entries are arbitrary.

3.2 The connected centerless simple group $G = PSL(n+1; \mathbf{C}) = SL(n+1; \mathbf{C})/\{\text{center}\}$ is transitive on the space consisting of points x and incident hyperplanes $u, ux = 0$, in P^n , as in 2.5. The isotropy subgroup P of the incident point and hyperplane

$$x_0 = {}^t(1, 0, \dots, 0), \quad u_0 = (0, \dots, 0, 1)$$

has exactly \mathfrak{p} for its Lie algebra. Hence, the homogeneous contact manifold which the construction of 2.10 gives is

$$\begin{aligned} G/P &= \text{space of incident points and hyperplanes in } P^n \\ &= \text{space of co-directions in complex } P^n. \end{aligned}$$

3.3 Let \mathfrak{m} be the $(2n-1)$ -dimensional supplement to \mathfrak{p} in \mathfrak{g} consisting of matrices of the form

$$\begin{bmatrix} 0 & & & & \\ & * & & & 0 \\ & \vdots & & & \\ & * & \cdots & * & 0 \end{bmatrix};$$

cf. 2.12. The product of any two matrices of \mathfrak{m} has a nonzero entry only in the n^{th} row and 0^{th} column; the product of any three is zero. Set

$$X = \begin{bmatrix} 0 & & & & \\ & x_1 & & & 0 \\ & \vdots & & & \\ & x_{n-1} & & & \\ x_n - \frac{1}{2} \sum p_i x_i & p_1 \cdots p_{n-1} & & & 0 \end{bmatrix},$$

where the summation is over $i = 1, 2, \dots, n-1$. X is in \mathfrak{m} and

$$\exp X = 1_{n+1} + X + \frac{1}{2} X^2 =$$

$$\begin{bmatrix} 1 & & & & \\ & x_1 & & & 0 \\ & \vdots & & & \\ & x_{n-1} & & & 0 \\ x_n & p_1 \cdots p_{n-1} & & & 1 \end{bmatrix},$$

and the $n0$ -entry is

$$\omega = dx_n - p_1 dx_1 - \dots - p_{n-1} dx_{n-1}.$$

This identifies the contact structure with the classical one as in 2.12.

3.5 The real contact structure on the $(2n-1)$ -dimensional space of co-directions in real projective space P^n is described by viewing all quantities in the foregoing discussion as being real. Especially, G_0 of 2.11 is the connected centerless group $PSL(n+1; \mathbf{R})$ consisting of real contact automorphisms.

4. HIGHER SPHERE GEOMETRY

4.1 In complex Euclidean space E^n , the equation

$$x_1'^2 + \dots + x_n'^2 - 2a_1 x_1' - \dots - 2a_n x_n' + C = 0$$

describes a sphere with center (a_1, \dots, a_n) and complex radius r given by

$$r^2 = a_1^2 + \dots + a_n^2 - C.$$

When $r \neq 0$, the two choices of sign for r is said to give two "orientations" to the sphere. Thus, the $n+2$ coordinates a_1, \dots, a_n, r, C , which are related by

$$a_1^2 + \dots + a_n^2 - r^2 - C = 0,$$

describe the space of oriented spheres in E^n [6, §25].

Introduce homogeneous coordinates by

$$a_i = \frac{\alpha_i}{v}, \quad r = \frac{\lambda}{v}, \quad C = \frac{\mu}{v},$$

$i = 1, 2, \dots, n$. Then the oriented spheres of E^n correspond to certain points of the quadric Ψ^{n+1} in P^{n+2} described by

$$\alpha_1^2 + \dots + \alpha_n^2 - \lambda^2 - \mu v = 0.$$

The sphere corresponding to the point $(\alpha_1, \dots, \alpha_n, \lambda, \mu, v)$ of Ψ^{n+1} is

$$v(x_1'^2 + \dots + x_n'^2) - 2\alpha_1 x_1' - \dots - 2\alpha_n x_n' + \mu = 0.$$

Ordinary spheres have finite nonzero radius r , so $v \neq 0$. For $v = 0$, we obtain oriented hyperplanes. For $\lambda = 0$, we obtain point spheres or hyperplanes with isotropic hyperplane coordinate vector; these carry no