

1. Introduction

Objektyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **25 (1979)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **20.09.2024**

Nutzungsbedingungen

Die ETH-Bibliothek ist Anbieterin der digitalisierten Zeitschriften. Sie besitzt keine Urheberrechte an den Inhalten der Zeitschriften. Die Rechte liegen in der Regel bei den Herausgebern.

Die auf der Plattform e-periodica veröffentlichten Dokumente stehen für nicht-kommerzielle Zwecke in Lehre und Forschung sowie für die private Nutzung frei zur Verfügung. Einzelne Dateien oder Ausdrucke aus diesem Angebot können zusammen mit diesen Nutzungsbedingungen und den korrekten Herkunftsbezeichnungen weitergegeben werden.

Das Veröffentlichen von Bildern in Print- und Online-Publikationen ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Die systematische Speicherung von Teilen des elektronischen Angebots auf anderen Servern bedarf ebenfalls des schriftlichen Einverständnisses der Rechteinhaber.

Haftungsausschluss

Alle Angaben erfolgen ohne Gewähr für Vollständigkeit oder Richtigkeit. Es wird keine Haftung übernommen für Schäden durch die Verwendung von Informationen aus diesem Online-Angebot oder durch das Fehlen von Informationen. Dies gilt auch für Inhalte Dritter, die über dieses Angebot zugänglich sind.

ON LIE'S HIGHER SPHERE GEOMETRY

by Jay P. FILLMORE

1. INTRODUCTION

In this paper we draw together two theories having their roots in the ideas of S. Lie over a century ago: *Lie's higher sphere geometry*, with its famous *line-sphere transformation* [5], and the theory of Lie groups, especially the description of a geometry by global Lie groups¹. Indeed, not until the 1960s, with the appearance of W. M. Boothby's description of homogeneous contact manifolds [1, 2] and with the appearance of parabolic subgroups, could this connection be established. One can now say, in terms of Lie groups, that the three-dimensional complex line and sphere geometries are isomorphic and that *the real line and sphere geometries are two distinct real forms of one geometry*. Furthermore, the line-sphere transformation gives explicitly the isomorphism of the complex forms.

In Section 2 we summarize the formulation of Boothby's theory for algebraic homogeneous contact manifolds and make some observations about their real forms. The classical contact manifolds of complex co-directions in projective space and of Lie's higher sphere geometry are described in general in terms of this theory in Sections 3 and 4. Finally, in Section 5, the connection with Plücker's line geometry in three dimensions is established, and the line-sphere transformation is brought into perspective. This introduction continues with an overview of F. Klein's formulation of Lie's theory [5, 6], Boothby's theory, and their connection.

To a line in complex projective space P^3 may be assigned Plücker coordinates

$$\begin{aligned}\xi_1 &= p_{12}, \quad \xi_2 = p_{31}, \quad \xi_3 = p_{23}, \\ \xi_4 &= p_{03}, \quad \xi_5 = p_{02}, \quad \xi_6 = p_{01},\end{aligned}$$

[6, §20]. These coordinates satisfy

$$\xi_1\xi_4 + \xi_2\xi_5 + \xi_3\xi_6 = 0,$$

¹ This description of Lie's higher sphere geometry in terms of Lie groups answers a question posed in 1965 by S. SASAKI [9, p. 173].

and hence lines in P^3 correspond to points of a quadric Ω^4 in P^5 . Two lines in P^3 intersect when their corresponding points on Ω^4 are conjugate. A surface element in P^3 , a point and incident plane, becomes the pencil of lines passing through the point and lying in the plane; this corresponds to a line lying in Ω^4 . The space of surface elements in P^3 thus corresponds to the space of lines in Ω^4 . The projectivities of P^5 which preserve the quadric Ω^4 permute the lines of Ω^4 and hence the surface elements of P^3 . Moreover, these projectivities preserve the condition, between two surface elements at infinitesimally adjacent points, that a point of one lies on the plane of the other; hence they are contact transformations of P^3 .

To a sphere

$$x^2 + y^2 + z^2 - 2ax - 2by - 2cz + C = 0$$

in complex Euclidean space E^3 , with center at $x = a, y = b, z = c$ and radius

$$r^2 = a^2 + b^2 + c^2 - C,$$

the sign of r corresponding to an "orientation", may be assigned homogeneous coordinates

$$a = \frac{\alpha}{\nu}, \quad b = \frac{\beta}{\nu}, \quad c = \frac{\gamma}{\nu}, \quad r = \frac{\lambda}{\nu}, \quad C = \frac{\mu}{\nu},$$

[6, §25]. These coordinates satisfy

$$\alpha^2 + \beta^2 + \gamma^2 - \lambda^2 - \mu\nu = 0,$$

and hence oriented spheres in E^3 correspond to certain points of a quadric Ψ^4 in P^5 ; if spheres which are points or planes or which have centers at infinity are included, all points of Ψ^4 are obtained. Two spheres in E^3 are tangent at a point, orientations taken into account, when their corresponding points on Ψ^4 are conjugate. An "oriented" surface element in E^3 , a point and incident oriented plane, becomes the pencil of spheres tangent to the plane at the point; this corresponds to a line lying in Ψ^4 . The space of oriented surface elements of E^3 thus corresponds to the space of lines in Ψ^4 . The projectivities of P^5 which preserve the quadric Ψ^4 permute the lines of Ψ^4 and hence the oriented surface elements of E^3 . Moreover, these projectivities are contact transformations of E^3 .

The line-sphere transformation, discovered by Lie, is given by

$$\begin{aligned} \xi_1 &= \alpha + \sqrt{-1}\beta, & \xi_4 &= \alpha - \sqrt{-1}\beta, \\ \xi_2 &= \gamma + \lambda, & \xi_5 &= \gamma - \lambda, \\ \xi_3 &= \mu, & \xi_6 &= -\nu, \end{aligned}$$

as formulated by Klein [6, §70]. This makes correspond points of the quadric Ω^4 of signature $(+++---)$ and points of the quadric Ψ^4 of signature $(++++--)$. Conjugate points correspond to conjugate points, and a line in one quadric corresponds to a line in the other. Thus, surface elements in P^3 correspond to oriented surface elements in E^3 and this correspondence is a “contact transformation”.

Now, classically a *contact transformation* in P^3 or E^3 is a transformation on the 5-dimensional space of surface elements which preserves, up to a non-vanishing multiple, a maximal rank Pfaffian form

$$\omega = dz - pdx - qdy,$$

[6, §63], where the coordinates x, y, z, p, q describe the surface element consisting of the plane

$$z' - z = p(x' - x) + q(y' - y)$$

at the point (x, y, z) . The condition $\omega = 0$, that at two infinitesimally adjacent points the point of one surface element lies on the plane of the other, is preserved by a contact transformation. The appropriate spaces for the line-sphere transformation are the 5-dimensional spaces of lines in Ω^4 and lines in Ψ^4 . Exhibiting the Pfaffian forms and examining the effect of the line-sphere transformation on them may be done systematically by observing that these spaces are *homogeneous*.

Boothby's description of compact homogeneous complex contact manifolds [1, 2; and 7, §2] constructs for each type of simple complex Lie algebra \mathfrak{g} : a connected centerless simple Lie groups G having Lie algebra \mathfrak{g} , a parabolic subgroup P of G , and a Pfaffian form ω on a principal C^* -bundle over G/P , so that G/P , with ω pulled down by local sections, is a compact complex contact manifold, homogeneous under the identity component G of the group of all its contact automorphisms. Every such contact manifold is so obtained uniquely up to isomorphism. This construction yields, for the classical simple Lie algebras:

- A_n projective cotangent bundle of P^n —the classical space of incident point-hyperplane pairs in P^n ,
- B_l and D_l space of lines in a quadric,
- C_l odd-dimensional projective space P^{2l+1} ,

[1, (7.1)]. The isomorphism $A_3 \simeq D_3$ arises from the description of surface elements in P^3 as lines in Ω^4 by Plücker coordinates. Since the

complex quadrics Ω^4 and Ψ^4 both have groups of projectivities of the type D_3 , the contact manifolds of line geometry and sphere geometry, when viewed as the spaces of lines in Ω^4 and Ψ^4 respectively, are necessarily *the same*, that is, isomorphic.

When Boothby's description of homogeneous contact manifolds is refined, using J. A. Wolf's theory of complex flag manifolds [8, Ch. I], to include their real forms, line geometry and sphere geometry are *no longer the same*, but, as was classically recognized [6, §25], are obtained from the real forms $PSO(3, 3; \mathbf{R})$ and $PSO(4, 2; \mathbf{R})$ of $PSO(3, 3; \mathbf{C})$ and $PSO(4, 2; \mathbf{C})$, where the quadratic forms defining these projective special orthogonal groups are those of the quadrics Ω^4 and Ψ^4 . Now, $PSO(3, 3; \mathbf{C})$ and $PSO(4, 2; \mathbf{C})$ are isomorphic, so the corresponding complex contact manifolds are isomorphic; in fact, these groups, are conjugate in $PSL(6; \mathbf{C})$ by the matrix of Klein's description of the line-sphere transformation. Viewed another way, $PSO(3, 3; \mathbf{R})$ and $PSO(4, 2; \mathbf{R})$ correspond to two real forms of $PSO(3, 3; \mathbf{C})$ defined by two complex conjugations. Consequently, *real line geometry and real sphere geometry are two distinct real forms of complex line geometry*. The line-sphere transformation then corresponds to an automorphism of $PSO(3, 3; \mathbf{C})$ connecting the two complex conjugations.

