

## 2. Integral representation theorems for linear functionals

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## 2. INTEGRAL REPRESENTATION THEOREMS FOR LINEAR FUNCTIONALS

Let  $A$  be a commutative Banach algebra over  $\mathbf{C}$  and let  $\Delta$  denote the locally compact space of regular maximal ideals of  $A$ . For each  $x \in A$  we use  $\hat{x}$  to denote the Gelfand-transform; i.e.,  $\hat{x}$  is the continuous mapping from  $\Delta$  to  $\mathbf{C}$  defined by the relations:

$$\hat{x}(m) = m(x) \quad \text{for } m \in \Delta.$$

By  $C_0(\Delta)$  we shall denote the algebra of all complex-valued continuous functions on  $\Delta$  which vanish at infinity. For any subset  $\mathcal{A} \subset A$  we shall use the notation  $\hat{\mathcal{A}}$  to denote the set  $\{\hat{x} : x \in \mathcal{A}\}$ . As usual  $\|\hat{x}\|_\infty$  denotes the supremum norm.

**THEOREM 1.** Let  $f$  be a linear form on the complex commutative Banach algebra  $A$  and let  $\mathcal{A}$  be a linear subspace of  $A$ . The following two statements are equivalent:

(1) There exists a constant  $M$  such that

$$|f(x)| \leq M \|\hat{x}\|_\infty \quad \text{for every } x \in \mathcal{A}.$$

(2) There exists a bounded complex Radon measure  $\mu$  on  $\Delta$  such that

$$f(x) = \int_\Delta \hat{x}(m) d\mu(m) \quad \text{for every } x \in \mathcal{A}.$$

*Proof.* The implication (2)  $\Rightarrow$  (1) is clear with  $M = \|\mu\|$ . We shall prove (1)  $\Rightarrow$  (2). Define a mapping  $L : \hat{\mathcal{A}} \rightarrow \mathbf{C}$  by

$$L(\hat{x}) = f(x).$$

It follows from (1) that  $L$  is well-defined, and that

$$|L(\hat{x})| \leq M \|\hat{x}\|_\infty \quad \text{for every } \hat{x} \in \hat{\mathcal{A}}$$

and so  $L$  is continuous with  $\|L\| \leq M$ . Using the Hahn-Banach Theorem we can extend  $L$  to a bounded linear form  $L_0$  on  $C_0(\Delta)$  and by the Riesz Representation Theorem we obtain the existence of a bounded complex Radon measure  $\mu$  on  $\Delta$  such that

$$\| \mu \| = \| L \| = \| L_0 \| \quad \text{and}$$

$$L_0(\varphi) = \int_{\Delta} \varphi(m) d\mu(m) \quad \text{for every } \varphi \in C_0(\Delta).$$

In particular

$$f(x) = L(\hat{x}) = L_0(\hat{x}) = \int_{\Delta} \hat{x}(m) d\mu(m) \quad \text{for every } x \in \mathcal{A}.$$

*Remark:* Suppose that  $A$  has an identity and that  $\hat{A}$  is closed under complex conjugation, then since  $\hat{A}$  contains constants and separates the points of  $\Delta$ , the Stone-Weierstraß Theorem implies that  $\hat{A}$  is dense in  $C(\Delta)$ , the algebra of all complex-valued continuous functions on the compact Hausdorff space  $\Delta$ . If we impose these additional conditions on  $A$  and if we take  $\mathcal{A} = A$  in Theorem 1, we can conclude that in this case the representing measure  $\mu$  is uniquely determined.

If the algebra  $A$  has a continuous involution, one can use Theorem 1 to derive an extended version of a theorem due to Raikov [10]. We proceed to describe the situation.

Let  $A$  be a complex commutative Banach algebra with an isometric involution  $*$  and a bounded approximate identity  $\{u_\lambda\}_{\lambda \in \Lambda}$  i.e., a net satisfying the following conditions:

$$\begin{aligned} \| u_\lambda \| &\leq 1 \quad \text{for each } \lambda \in \Lambda, \\ \| u_\lambda x - x \| &\rightarrow 0 \quad \text{for each } x \in A. \end{aligned}$$

A continuous *positive* functional on  $A$  is an element  $f \in A'$  such that  $f(x^*x) \geq 0$  for every  $x \in A$ . If  $f$  is a continuous positive functional on  $A$  then the Cauchy-Schwarz inequality is valid (Dixmier [8, p. 23]) and this implies the following facts:

$$\begin{aligned} f(u_\lambda) &\rightarrow \| f \| \\ | f(x) |^2 &\leq \| f \| f(x^*x) \quad \text{for every } x \in A. \end{aligned}$$

If the involution is *symmetric*, which means  $(x^*)^\wedge = \overline{x}$  for every  $x \in A$  or, equivalently, that every  $m \in \Delta$  is a *positive* linear functional, then by modifying a classical method of Gelfand-Raikov-Silov [10; p. 62] one can prove that

$$| f(x) | \leq \| f \| \| \hat{x} \|_\infty \quad \text{for every } x \in A.$$

As a corollary to Theorem 1 and the above discussion we obtain the following extended theorem of Raikov [10; p. 64], see also Bucy-Maltese [4]):

**THEOREM 2.** Let  $A$  be a commutative Banach algebra with an isometric involution which is symmetric. Suppose that  $A$  has a bounded approximate identity and let  $f \in A'$  be a continuous positive functional. Then there exists a unique positive Radon measure  $\mu$  on  $\Delta$  such that  $\|\mu\| = \|f\|$  and

$$f(x) = \int_A \hat{x}(m) d\mu(m) \quad \text{for every } x \in A.$$

*Proof.* From the above remarks we know that

$$|f(x)| \leq \|f\| \|\hat{x}\|_\infty \quad \text{for every } x \in A.$$

By Theorem 1 there exists a complex Radon measure  $\mu$  on  $\Delta$  such that  $\|\mu\| \leq \|f\|$  and

$$f(x) = \int_A \hat{x}(m) d\mu(m) \quad \text{for every } x \in A.$$

This formula implies

$$|f(x)| \leq \|\hat{x}\|_\infty \|\mu\| \leq \|x\| \|\mu\| \quad \text{for every } x \in A,$$

so that  $\|f\| \leq \|\mu\|$  and hence  $\|f\| = \|\mu\|$ .

Since  $\hat{A}$  is a self-adjoint subalgebra of  $C_0(\Delta)$  which separates points and for each  $m \in \Delta$  contains a function  $\hat{x}$  such that  $\hat{x}(m) \neq 0$  (in fact <sup>1)</sup> there exists an element  $u_\beta$  of the approximate identity such that  $\hat{u}_\beta(m) \neq 0$ ), the Stone-Weierstraß Theorem implies the uniqueness of the measure  $\mu$ . The positivity of  $\mu$  also follows from the fact that  $\hat{A}$  is dense in  $C_0(\Delta)$ . In fact if  $p$  is a non-negative function in  $C_0(\Delta)$ , then  $p = |q|^2$  for some  $q \in C_0(\Delta)$ . Choose a sequence  $\{x_n\}$  in  $A$  such that

$$\hat{x}_n \rightarrow q.$$

<sup>1)</sup> If  $m \in \Delta$ , then  $\|m\| \neq 0$  and by the assumption of symmetry  $m$  is a positive functional. Therefore, as mentioned above,  $\|m\| = \lim_{\alpha} m(u_\alpha)$  so that there must exist some  $u_\beta$  of the approximate identity such that  $m(u_\beta) \neq 0$ .

Since  $(x_n^*)^\wedge = \overline{x_n}$  it follows that  $(x_n^*)^\wedge \rightarrow \bar{q}$  and hence

$$(x_n x_n^*)^\wedge \rightarrow |q|^2 = p.$$

This implies

$$\begin{aligned} \int_{\Delta} p(m) d\mu(m) &= \lim_n \int_{\Delta} (x_n^* x_n)^\wedge(m) d\mu(m) \\ &= \lim_n f(x_n^* x_n) \geq 0, \end{aligned}$$

so that  $\mu$  is a positive measure and this completes the proof.

If  $A$  has an identity, as is the case in Raikov's original version, the above proof can be somewhat simplified.

**THEOREM 3 (Raikov).** Let  $A$  be a complex commutative Banach algebra with an identity  $e$  and with an isometric involution which is symmetric. If  $f$  is a continuous positive functional on  $A$ , then there exists a unique positive Radon measure  $\mu$  on  $\Delta$  such that  $\|\mu\| = \|f\|$  and

$$f(x) = \int_{\Delta} \hat{x}(m) d\mu(m) \quad \text{for every } x \in A.$$

*Proof.* As above we know that

$$|f(x)| \leq \|f\| \|\hat{x}\|_{\infty} \quad \text{for every } x \in A.$$

From Theorem 1 there exists a complex Radon measure  $\mu$  on  $\Delta$  such that  $\|\mu\| \leq \|f\|$  and

$$f(x) = \int_{\Delta} \hat{x}(m) d\mu(m) \quad \text{for every } x \in A.$$

Hence  $\|\mu\| \leq \|f\| = f(e) = \mu(1) \leq \|\mu\|$  so that  $\mu(1) = \|\mu\|$  which is enough to imply that  $\mu$  is positive. The uniqueness of  $\mu$  follows as in the Remark to Theorem 1.

### 3. APPLICATIONS OF THE INTEGRAL REPRESENTATION THEOREMS

*Application 1 (Bochner's Theorem).* Let  $G$  be a locally compact abelian group and let  $\hat{G}$  denote the (locally compact) character group. Denote