2. Integral representation theorems for linear functionals

Let A be a commutative Banach algebra over $\mathbf{C}$ and let Δ denote the locally compact space of regular maximal ideals of A. For each $x \in A$ we use $\hat{x}$ to denote the Gelfand-transform; i.e., $\hat{x}$ is the continuous mapping from Δ to $\mathbf{C}$ defined by the relations:

$$\hat{x}(m) = m(x) \quad \text{for} \quad m \in \Delta .$$

By $C_0(\Delta)$ we shall denote the algebra of all complex-valued continuous functions on Δ which vanish at infinity. For any subset $\mathscr{A} \subset A$ we shall use the notation $\hat{\mathscr{A}}$ to denote the set $\{ \hat{x} : x \in \mathscr{A} \}$. As usual $\| \hat{x} \|_\infty$ denotes the supremum norm.

THEOREM 1. Let f be a linear form on the complex commutative Banach algebra A and let $\mathscr{A}$ be a linear subspace of A. The following two statements are equivalent:

(1) There exists a constant M such that

$$| f(x) | \leqslant M \| \hat{x} \|_\infty \quad \text{for every } x \in \mathscr{A} .$$

(2) There exists a bounded complex Radon measure μ on Δ such that

$$f(x) = \int_\Delta \hat{x}(m)\, d\mu(m) \quad \text{for every} \quad x \in \mathscr{A}.$$

Proof. The implication (2) ⇒ (1) is clear with $M = \| \mu \|$. We shall prove (1) ⇒ (2). Define a mapping $L : \hat{\mathscr{A}} \to \mathbf{C}$ by

$$L(\hat{x}) = f(x) .$$

It follows from (1) that L is well-defined, and that

$$| L(\hat{x}) | \leqslant M \| \hat{x} \|_\infty \quad \text{for every} \quad \hat{x} \in \hat{\mathscr{A}}$$

and so L is continuous with $\| L \| \leqslant M$. Using the Hahn-Banach Theorem we can extend L to a bounded linear form L_0 on $C_0(\Delta)$ and by the Riesz Representation Theorem we obtain the existence of a bounded complex Radon measure μ on Δ such that

$$\| \mu \| = \| L \| = \| L_0 \| \quad \text{and}$$

$$L_0 (\varphi) = \int_\Delta \varphi (m) \, d\mu (m) \quad \text{for every} \quad \varphi \in C_0 (\Delta) .$$

In particular

$$f (x) = L (\hat{x}) = L_0 (\hat{x}) = \int_\Delta \hat{x} (m) \, d\mu (m) \quad \text{for every} \quad x \in \mathscr{A} .$$

Remark: Suppose that A has an identity and that $\hat{A}$ is closed under complex conjugation, then since $\hat{A}$ contains constants and separates the points of Δ, the Stone-Weierstraß Theorem implies that $\hat{A}$ is dense in $\mathbf{C} (\Delta)$, the algebra of all complex-valued continuous functions on the compact Hausdorff space Δ. If we impose these additional conditions on A and if we take $\mathscr{A} = A$ in Theorem 1, we can conclude that in this case the representing measure μ is uniquely determined.

If the algebra A has a continuous involution, one can use Theorem 1 to derive an extended version of a theorem due to Raikov [10]. We proceed to describe the situation.

Let A be a complex commutative Banach algebra with an isometric involution * and a bounded approximate identity $\{ u_\lambda \}_{\lambda \in \Lambda}$ i.e., a net satisfying the following conditions:

$$\| u_\lambda \| \leqslant 1 \quad \text{for each} \quad \lambda \in \Lambda ,$$
$$\| u_\lambda x - x \| \to 0 \quad \text{for each} \quad x \in A .$$

A continuous *positive* functional on A is an element $f \in A'$ such that $f (x^* x) \geqslant 0$ for every $x \in A$. If f is a continuous positive functional on A then the Cauchy-Schwarz inequality is valid (Dixmier [8, p. 23]) and this implies the following facts:

$$f(u_\lambda) \to \| f \|$$
$$| f (x) |^2 \leqslant \| f \| \, f (x^* x) \quad \text{for every} \quad x \in A .$$

If the involution is *symmetric*, which means $(x^*)^\wedge = \overline{\hat{x}}$ for every $x \in A$ or, equivalently, that every $m \in \Delta$ is a *positive* linear functional, then by modifying a classical method of Gelfand-Raikov-Silov [10; p. 62] one can prove that

$$| f (x) | \leqslant \| f \| \, \| \hat{x} \|_\infty \quad \text{for every} \quad x \in A .$$

As a corollary to Theorem 1 and the above discussion we obtain the following extended theorem of Raikov [10; p. 64], see also Bucy-Maltese [4]):

THEOREM 2. Let A be a commutative Banach algebra with an isometric involution which is symmetric. Suppose that A has a bounded approximate identity and let $f \in A'$ be a continuous positive functional. Then there exists a unique positive Radon measure μ on Δ such that $\| \mu \| = \| f \|$ and

$$f(x) = \int_{\Delta} \hat{x}(m) \, d\mu(m) \quad \text{for every} \quad x \in A .$$

Proof. From the above remarks we know that

$$| f(x) | \leqslant \| f \| \; \| \hat{x} \|_{\infty} \quad \text{for every} \quad x \in A .$$

By Theorem 1 there exists a complex Radon measure μ on Δ such that $\| \mu \| \leqslant \| f \|$ and

$$f(x) = \int_{\Delta} \hat{x}(m) \, d\mu(m) \quad \text{for every} \quad x \in A .$$

This formula implies

$$| f(x) | \leqslant \| \hat{x} \|_{\infty} \; \| \mu \| \leqslant \| x \| \; \| \mu \| \quad \text{for every} \quad x \in A ,$$

so that $\| f \| \leqslant \| \mu \|$ and hence $\| f \| = \| \mu \|$.

Since $\hat{A}$ is a self-adjoint subalgebra of $\mathbf{C}_0(\Delta)$ which separates points and for each $m \in \Delta$ contains a function $\hat{x}$ such that $\hat{x}(m) \neq 0$ (in fact [1]) there exists an element u_β of the approximate identity such that $\hat{u}_\beta(m) \neq 0$), the Stone-Weierstraß Theorem implies the uniqueness of the measure μ. The positivity of μ also follows from the fact that $\hat{A}$ is dense in $\mathbf{C}_0(\Delta)$. In fact if p is a non-negative function in $\mathbf{C}_0(\Delta)$, then $p = | q |^2$ for some $q \in \mathbf{C}_0(\Delta)$. Choose a sequence $\{ x_n \}$ in A such that

$$\hat{x}_n \to q .$$

[1]) If $m \in \Delta$, then $\| m \| \neq 0$ and by the assumption of symmetry m is a positive functional. Therefore, as mentioned above, $\| m \| = \lim_{\alpha} m(u_\alpha)$ so that there must exist some u_β of the approximate identity such that $m(u_\beta) \neq 0$.

Since $(x_n^*)^\wedge = \overline{\hat{x}}_n$ it follows that $(x_n^*)^\wedge \to \bar{q}$ and hence

$$(x_n x_n^*)^\wedge \to |q|^2 = p .$$

This implies

$$\begin{aligned} \int_\Delta p(m)\, d\mu(m) &= \lim_n \int_\Delta (x_n^* x_n)^\wedge)(m)\, d\mu(m) \\ &= \lim_n f(x_n^* x_n) \geqslant 0 , \end{aligned}$$

so that μ is a positive measure and this completes the proof.

If A has an identity, as is the case in Raikov's original version, the above proof can be somewhat simplified.

THEOREM 3 (Raikov). Let A be a complex commutative Banach algebra with an identity e and with an isometric involution which is symmetric. If f is a continuous positive functional on A, then there exists a unique positive Radon measure μ on Δ such that $\|\mu\| = \|f\|$ and

$$f(x) = \int_\Delta \hat{x}(m)\, d\mu(m) \quad \text{for every} \quad x \in A .$$

Proof. As above we know that

$$|f(x)| \leqslant \|f\| \, \|\hat{x}\|_\infty \quad \text{for every} \quad x \in A .$$

From Theorem 1 there exists a complex Radon measure μ on Δ such that $\|\mu\| \leqslant \|f\|$ and

$$f(x) = \int_\Delta \hat{x}(m)\, d\mu(m) \quad \text{for every} \quad x \in A .$$

Hence $\|\mu\| \leqslant \|f\| = f(e) = \mu(1) \leqslant \|\mu\|$ so that $\mu(1) = \|\mu\|$ which is enough to imply that μ is positive. The uniqueness of μ follows as in the Remark to Theorem 1.

3. APPLICATIONS OF THE INTEGRAL REPRESENTATION THEOREMS

Application 1 (Bochner's Theorem). Let G be a locally compact abelian group and let $\hat{G}$ denote the (locally compact) character group. Denote