3. Quasicircles

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3. Quasicircles

3.1 *Definition.* A Jordan curve is the image of a circle under a homeomorphism of the plane. If the homeomorphism can be taken to be a K-quasiconformal mapping, the Jordan curve is called a *K-quasicircle.*

For a later application, we need the following result.

Lemma 3.1. *A K-quasicircle is the image of the real axis under a quasiconformal mapping of the plane which is conformal in the upper half-plane and K^2-quasiconformal in the lower half-plane.*

Proof: Let C be a K-quasicircle. Then there is a K-quasiconformal mapping w of the plane which carries the real axis onto C. Let μ denote the complex dilatation of w. By the existence theorem for Beltrami equations, there is a quasiconformal self-mapping h of the upper half-plane with complex dilatation μ. If h is extended to the lower half-plane by reflection in the real axis, we obtain a K-quasiconformal mapping of the plane. Then $w \circ h^{-1}$ has the desired properties: by the uniqueness theorem for Beltrami equations, it is conformal in the upper half-plane, and as a composition of two K-quasiconformal mappings it is K^2-quasiconformal in the lower half-plane.

The notion of a quasicircle was introduced by Pfluger [15]; he arrived at these curves, which he called "kreisähnlich", in connection with a sewing problem for Riemann surfaces. Pfluger proved that a quasicircle, while always of zero area, need not be rectifiable. Later, Gehring and Väisälä [4] showed that the Hausdorff dimension of a quasicircle is always < 2 but can take any value λ, $1 \leqslant \lambda < 2$.

3.2 *Geometric characterization.* The first systematic study of quasicircles is Tienari's thesis [16]. His results were soon overshadowed by Ahlfors [1], who gave an amazingly simple geometric characterization of quasicircles: A Jordan curve passing through ∞ is a quasicircle if and only if for any of its three successive finite points z_1, z_2, z_3, the ratio $|z_1 - z_2| : |z_1 - z_3|$ is uniformly bounded.

The condition of Ahlfors can be modified in various ways. Let $U(z, r) = \{w \,\|\, w - z| < r\}$ and let $\mathrm{cl}\,U$ denote the closure of U. A set E of the extended plane is *b-locally connected* if the following two conditions hold for every finite z and every $r > 0$:

1° Any two points of the set $E \cap \mathrm{cl}\, U(z, r)$ can be joined by an arc lying in $E \cap \mathrm{cl}\, U(z, br)$.

2° Any two points of the set $E - U(z, r)$ can be joined by an arc lying in $E - U(z, r/b)$.

The following result has recently been proved by Gehring [2]:

LEMMA 3.2. *Let the set* C *contain at least two points and bound a simply connected domain* A. *If* A *is b-locally connected, then* C *is a* $c(b)$*-quasicircle, where* $c(b)$ *depends only on* b.

3.3 *Quasiconformal reflection.* Let C be a Jordan curve bounding the domains A and B. A sense-reversing K-quasiconformal mapping $\varphi\colon A \to B$ is a *K-quasiconformal reflection* in C if φ leaves every point of C invariant.

It is not difficult to prove that C admits a quasiconformal reflection if and only if C is a quasicircle. It follows that a quasiconformal mapping $f\colon A \to B$ between domains A and B bounded by quasicircles can be extended to a quasiconformal mapping of the plane. In fact, if φ and ψ are quasiconformal reflections in the boundaries ∂A and ∂B, such that φ is defined outside A and ψ in B, then $\psi \circ f \circ \varphi$ extends f quasiconformally.

A quasicircle always admits quasiconformal reflections which are continuously differentiable or even real-analytic. For a K-quasicircle passing through ∞, a reflection φ exists such that $|d\varphi(z)| / |dz|$ is bounded by a constant depending only on K.

For more details of the properties of quasicircles we refer to [10].

4. Deviation of a domain from a disc

4.1 *Schwarzian derivative.* Let f be a locally injective meromorphic function in a simply connected domain A. At finite points of A which are not poles of f, the *Schwarzian derivative* S_f of f is defined by

$$S_f = (f''/f')' - \frac{1}{2}(f''/f')^2,$$

and the definition is extended to ∞ and to the poles of f by means of inversion.

The Schwarzian derivative is holomorphic in A. Conversely, every function which is holomorphic in A is the Schwarzian of some f. The Schwarzian vanishes identically if and only if f is a Möbius transformation.