3. Spirals

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We give an argument to show the equivalence of Theorems 1 and 3. Suppose first that Theorem 1 holds. Then there exists a $\varphi \in S$ and a $\delta > 0$ such that $\| \psi - \varphi \| > \delta$ for all $\psi \in T$. Choose g conformal in L with $S_g = \varphi$, let $D = g(L)$ and suppose that f is conformal in D with $\| S_f \|_D \leqslant \delta$. Then $h = f \circ g$ is conformal in L,

$$(2) \qquad S_h = (S_f \circ g)(g')^2 + S_g$$

by the composition law for the Schwarzian derivative, and hence $\psi = S_h \in S$ with

$$\| \psi - \varphi \| = \| S_h - S_g \|_L = \| S_f \|_D \leqslant \delta .$$

Thus $\psi \notin T$, h does not have a quasiconformal extension to $\overline{\mathbf{C}}$, and $\partial f(D) = \partial h(L)$ is not a quasiconformal circle. Hence Theorem 3 holds.

Suppose next that Theorem 3 holds, let $\varphi = S_g$ where g is any conformal mapping of L onto D, and choose any $\psi \in S$ with $\| \psi - \varphi \| \leqslant \delta$. Then $\psi = S_h$ where h is conformal in L, $f = h \circ g^{-1}$ is conformal in D and from (2) we obtain

$$\| S_f \|_D = \| S_h - S_g \|_L = \| \psi - \varphi \| \leqslant \delta .$$

Hence $\partial h(L) = \partial f(D)$ is not a quasiconformal circle, h does not have a quasiconformal extension to $\overline{\mathbf{C}}$ and $\psi \notin T$. Thus the distance from φ to T is at least δ and Theorem 1 holds.

A simple modification of the above argument yields the equivalence of Theorems 2 and 4.

Theorems 1 and 3 are immediate consequences of the following result.

THEOREM 5. *There exists a simply connected domain D and a positive constant δ such that $f(D)$ is not a Jordan domain whenever f is conformal in D with $\| S_f \|_D \leqslant \delta$.*

3. SPIRALS

The proof of Theorem 5 is based on two results for a class of spirals.

DEFINITION. *We say that an open arc α in $\mathbf{C}$ is a b-spiral from z_1 onto z_2 if α has the representation*

$$z = (z_1 - z_2)\, r(t)\, e^{it} + z_2 , \quad 0 < t < \infty ,$$

where $r(t)$ is positive and continuous with

$$\lim_{t \to 0} r(t) = 1, \quad \lim_{t \to \infty} r(t) = 0,$$

and where $r(t_1) \leqslant b\, r(t_2)$ *for all* t_1, t_2 *with* $|t_1 - t_2| \leqslant 2\pi$.

When a is a positive constant, the arc

$$\alpha = \{z = e^{(-a+i)t} : 0 < t < \infty\}$$

is an $e^{2\pi a}$-spiral from 1 onto 0. Moreover,

$$k(z)\,|z| = c, \quad \frac{dk}{ds}(z)\,|z|^2 = d \tag{3}$$

for all $z \in \alpha$, where c and d are positive constants with $d = ac^2$, and where k and s denote the curvature and arclength of α.

The first result we need shows that a curvature condition, similar to (3), is sufficient to guarantee that an open arc is a b-spiral.

LEMMA 1. *Suppose that* α *is an analytic open arc with* 1 *and* 0 *as endpoints, and suppose that*

$$c_1 \leqslant k(z)\,|z| \leqslant c_2, \quad d_1 \leqslant \frac{dk}{ds}(z)\,|z|^2 \leqslant d_2 \tag{4}$$

for all $z \in \alpha$, *where* c_1, c_2, d_1, d_2 *are positive constants with* $4\pi d_2 < c_1^2$. *Then* α *is a rectifiable* b*-spiral from* 1 *onto* 0 *where*

$$b = \frac{c_1\, c_2}{c_1^2 - 4\pi d_2}.$$

The second result we require implies that when b is near 1, the points onto which two disjoint b-spirals converge either coincide or are separated by a distance greater than $\dfrac{1}{2b^2}$ times the diameter of the smaller spiral.

LEMMA 2. *Suppose that* α *and* β *are disjoint* b*-spirals from* z_1 *onto* z_2 *and from* w_1 *onto* w_2, *respectively. If* $b \in (1, 2)$, *then either* $z_2 = w_2$ *or*

$$|z_2 - w_2| > \frac{1}{b} \min(|z_1 - z_2|, |w_1 - w_2|).$$