# 2. Reformulations in the plane

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The set $T$ is called the universal Teichmüller space. An important result due to Ahlfors and Bers shows that each Teichmüller space of a Riemann surface $R$ or of a Fuchsian group $G$ has a canonical embedding in the space $T$. See, for example, [3].

It is natural to ask if there exist relations, other than (1), between $S$ and $T$ as subsets of $B_2$. Compactness results for conformal mappings show that $S$ is closed in $B_2$. Hence Bers asked in [2] and [3] if one can characterize $S$ in terms of $T$ as follows.

QUESTION. *Is $S$ the closure of $T$?*

We shall answer this question in the negative by sketching a proof for the following result.

THEOREM 1. *There exists a $\varphi$ in $S$ which does not lie in the closure of $T$.*

On the other hand, we have the following characterization of $T$ in terms of $S$. See [4].

THEOREM 2. *$T$ is the interior of $S$.*

## 2. Reformulations in the plane

A set $E \subset \overline{\mathbf{C}}$ is said to be a *quasiconformal circle* if there exists a quasiconformal mapping $f$ defined in $\overline{\mathbf{C}}$ which maps the unit circle $\{z: |z| = 1\}$ onto $E$.

Theorems 1 and 2 are then respectively equivalent to the following two results on plane domains $D$.

THEOREM 3. *There exists a simply connected domain $D$ and a positive constant $\delta$ such that $f(D)$ is not bounded by a quasiconformal circle whenever $f$ is conformal in $D$ with $\| S_f \|_D \leqslant \delta$.*

THEOREM 4. *A simply connected domain $D$ is bounded by a quasiconformal circle if and only if there exists a positive constant $\delta$ such that $f$ is univalent in $D$ whenever $f$ is meromorphic in $D$ with $\| S_f \|_D \leqslant \delta$.*

We give an argument to show the equivalence of Theorems 1 and 3. Suppose first that Theorem 1 holds. Then there exists a $\varphi \in S$ and a $\delta > 0$ such that $\| \psi - \varphi \| > \delta$ for all $\psi \in T$. Choose $g$ conformal in $L$ with $S_g = \varphi$, let $D = g(L)$ and suppose that $f$ is conformal in $D$ with $\| S_f \|_D \leqslant \delta$. Then $h = f \circ g$ is conformal in $L$,

$$S_h = (S_f \circ g)(g')^2 + S_g \tag{2}$$

by the composition law for the Schwarzian derivative, and hence $\psi = S_h \in S$ with

$$\| \psi - \varphi \| = \| S_h - S_g \|_L = \| S_f \|_D \leqslant \delta .$$

Thus $\psi \notin T$, $h$ does not have a quasiconformal extension to $\overline{\mathbf{C}}$, and $\partial f(D) = \partial h(L)$ is not a quasiconformal circle. Hence Theorem 3 holds.

Suppose next that Theorem 3 holds, let $\varphi = S_g$ where $g$ is any conformal mapping of $L$ onto $D$, and choose any $\psi \in S$ with $\| \psi - \varphi \| \leqslant \delta$. Then $\psi = S_h$ where $h$ is conformal in $L$, $f = h \circ g^{-1}$ is conformal in $D$ and from (2) we obtain

$$\| S_f \|_D = \| S_h - S_g \|_L = \| \psi - \varphi \| \leqslant \delta .$$

Hence $\partial h(L) = \partial f(D)$ is not a quasiconformal circle, $h$ does not have a quasiconformal extension to $\overline{\mathbf{C}}$ and $\psi \notin T$. Thus the distance from $\varphi$ to $T$ is at least $\delta$ and Theorem 1 holds.

A simple modification of the above argument yields the equivalence of Theorems 2 and 4.

Theorems 1 and 3 are immediate consequences of the following result.

THEOREM 5. *There exists a simply connected domain $D$ and a positive constant $\delta$ such that $f(D)$ is not a Jordan domain whenever $f$ is conformal in $D$ with $\| S_f \|_D \leqslant \delta$.*

## 3. SPIRALS

The proof of Theorem 5 is based on two results for a class of spirals.

DEFINITION. *We say that an open arc $\alpha$ in $\mathbf{C}$ is a b-spiral from $z_1$ onto $z_2$ if $\alpha$ has the representation*

$$z = (z_1 - z_2)\, r(t)\, e^{it} + z_2 , \quad 0 < t < \infty ,$$

*where $r(t)$ is positive and continuous with*