9. Construction of a model for $C^\star (L_{M,N})$

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **24 (1978)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **19.09.2024**

To have a model for the homomorphism induced by the inclusion of $\mathfrak{a}_{n,p}$ in $\mathfrak{a}_n$, we have the commutative diagramm

$$\begin{array}{ccc} C^*(\mathfrak{a}_n) & \longrightarrow & C^*(\mathfrak{a}_{n,p}) \\ \uparrow & & \uparrow \\ W U_n & \longleftarrow & W U_{n,p} \end{array}$$

where the second horizontal map sends h_i on $h_i' + h_i''$ and c_i on $c_i' + c_i''$ (by convention, h_i' or h_i'' is zero for $i > p$ or $i > n\text{-}p$, idem for c_i' and c_i''). Note that the natural map of theorem 1 should map the c_i' s and c_i'' not on the usual Chern classes defined by the connection but on the polynomials in Chern classes corresponding to $\sum x_k^i$, the Chern classes being the elementary symmetric functions in the formal variables x_k. These horizontal maps are also models for an inclusion of $F_{n,p}$ in F_n.

We consider again the bundle E over M associated to the tangent bundle of M and with fiber F_n. Its restriction above N contains a subbundle E' with fiber $F_{n,p}$.

THEOREM. *$C^*(L_{M,N})$ is a model for the space $\Gamma_{M,N}$ of continuous sections of the bundle E whose restriction to N have values in the subbundle E'.*

To make explicit computations, we construct a model for $\Gamma_{M,N}$, which will be finite dimensional in each degree when M and N have finite dimensional models. This is the purpose of the next paragraph.

9. CONSTRUCTION OF A MODEL FOR $C^*(L_{M,N})$

Consider the commutative diagramm of Lie algebras

$$\begin{array}{ccc} L_{M,N} & \longrightarrow & L_M \\ \downarrow & & \downarrow \\ L'_{M,N} & \longrightarrow & L'_M \end{array}$$

where L'_M and $L'_{M,N}$ are the quotients of L_M and $L_{M,N}$ by the subalgebra L_M^0 of vector fields on M whose infinite jet vanish at points of N.

The corresponding geometric diagramm is

$$\begin{array}{ccc} \Gamma_{M,N} & \longrightarrow & \Gamma_M \\ \Big\downarrow & & \Big\downarrow \pi \\ \Gamma'_{M,N} & \longrightarrow & \Gamma'_M \end{array}$$

where Γ'_M denotes the space of sections of E restricted to N and $\Gamma'_{M,N}$ the space of sections of E'. The vertical maps associate to a section its restriction above N.

π is a fibration and $\Gamma_{M,N}$ is the fiber product of Γ_M and $\Gamma'_{M,N}$ over Γ'_M.

The spectral sequence of the fibration $\Gamma_{M,N} \to \Gamma'_{M,N}$ will correspond to the Hochschild-Serre spectral sequence [16] associated to the ideal L^0_M in $L_{M,N}$ (using continuous cochains). The DG-algebra $C^*(L^0_M)$ will be a model for the fiber.

We assume that we can represent the inclusion of N *in* M *by a surjection* $r \colon A \to B$ *of DG-algebras which are finite dimensional and such that* $A^i = 0$ *for* $i > n = \dim M$ *and* $B^i = 0$ *for* $i > p = \dim N$.

This is possible in particular if M and N are simply connected with finite dimensional real cohomology.

Let $a_1, \ldots, a_s, b_1, \ldots, b_t$ be a basis of A such that the $a'_i s$ form a basis of the kernel $\bar{A}$ of r. Hence the $r(b_j)$'s form a basis of B.

Let $\Lambda(x_\alpha)$ (resp. $\Lambda(y_\lambda)$) be a minimal model for F_n (resp. $F_{n,p}$), or equivalently of WU_n (resp. $WU_{n,p}$). Then the bundle E (resp. E') has a minimal model of the form $A \otimes \Lambda(x_\alpha)$ (resp. $B \otimes \Lambda(y_\lambda)$), where the differential is twisted by terms depending on the choice of representatives for the Pontrjagin classes of M (cf. [13]).

A model for $\Gamma_{M,N}$ will be the free algebra $\Lambda(x^i{}_\alpha, y^j{}_\lambda)$ on generators x^i_α, $i = 1, \ldots, s$, and $y^j{}_\lambda$, $j = 1, \ldots, t$, $\deg x^i_\alpha = \deg x - \deg a_i$, $\deg y^j{}_\lambda = \deg y_\lambda - \deg b_j$.

To get the differential, we proceed as follows. Recall that a model for Γ_M is the algebra $\Lambda(x^i{}_\alpha, z^j{}_\alpha)$, $\deg z^j_\alpha = \deg x_\alpha - \deg b_j$, with a suitable differential (cf. [18], [13] or § 5 with G the identity). Also models for $\Gamma'_{M,N}$ and Γ'_M are of the form $\Lambda(y^j{}_\lambda)$ and $\Lambda(z^i{}_\alpha)$, resp. with suitable differentials. One has DG-algebra maps

$$\Lambda(z^j{}_\alpha) \to \Lambda(x^j{}_\alpha, z^j{}_\alpha)$$
$$\Lambda(z^j{}_\alpha) \to \Lambda(y^j{}_\lambda)$$

which are models for the maps $\Gamma_M \to \Gamma'_M$ and $\Gamma'_{M,N} \to \Gamma'_M$. The first one is obvious and the second one is completely characterized by the map $WU_n \to WU_{n,p}$.

Now we get the differential on $\Lambda(x^j_\alpha, y^j_\lambda)$ by considering this algebra as the tensor product over $\Lambda(z^j_\alpha)$ of $\Lambda(x^j_\alpha, z^j_\lambda)$ with $\Lambda(y^j_\lambda)$.

One can make a similar construction using for A and B the DG-algebras Ω_M and Ω_N of differential forms on M and N. Of course one has to work again in more intrisic terms and use the C^∞-topology on Ω_M and Ω_N (compare with [13]). In this way one gets a DG-algebra which is also a model for $\Gamma_{M,N}$ (in fact one proves directly that it is a model for the DG-algebra constructed above), with a map in $C^*(L_{M,N})$ inducing an isomorphism in cohomology.

Summing up, we get the following result.

THEOREM. *Assume that the inclusion of N in M has a model which is a surjection of finite dimensional DG-algebras. One can construct explicitely a model for $C^*(L_{M,N})$ which is finite dimensional in each degree.*

Example. Suppose that M is the disk D^2 and N its boundary $\partial D^2 = S^1$. As the inclusion of $F_{2,1}$ in F_2 is homotopically trivial (equivalently the morphism $WU_2 \to WU_{2,1}$ is homotopic to zero), the bundle $\Gamma_{M,N} \to \Gamma'_{M,N}$ is trivial. WU_2 is a model for $S^5 \vee S^5 \vee S^7 \vee S^8 \vee S^8$ and $WU_{2,1}$ for $S^3 \vee S^3 \vee S^3 \vee S^4 \vee S^4$.

Hence $C^*(L_{D^2, \partial D^2})$ is a model for the space which is the product of the space of maps of S^1 in $S^3 \vee S^3 \vee S^3 \vee S^4 \vee S^4$ with the second loop space of $S^5 \vee S^5 \vee S^7 \vee S^8 \vee S^8$.

One can write down quite explicitely the minimal model for that space, but it is harder to compute the cohomology of the first factor. It has an infinite number of multiplicative generators.

10. Some other problems

1. As coefficient for the Gelfand-Fuks cochains, one might consider, instead of the field R with the trivial action of L_M, a topological L_M-algebra A. The problem is to find a model for the DG-algebra $C^*(L_M, A)$ of continuous multilinear alternate forms on L_M with values in A. The differential is defined by the usual formula involving the action of L_M on A.