

# 3. Elimination theory

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$I_0 = (0)$  and  $\mathfrak{S}_0 = \mathfrak{S}$  and define inductively  $r_d, I_d$  and  $\mathfrak{S}_d$  as follows. For  $d \geq 0$ , let  $r_{d+1}$  be equal to the maximum of the dimensions of  $I \cap R_{d+1}$  for  $I$  running over  $\mathfrak{S}_d$ , let  $I_{d+1}$  be any ideal in  $\mathfrak{S}_d$  such that  $\dim(I_{d+1} \cap R_{d+1}) = r_{d+1}$  and let  $\mathfrak{S}_{d+1}$  be the set of ideals  $I$  in  $\mathfrak{S}_d$  such that  $I \cap R_{d+1} = I_{d+1} \cap R_{d+1}$ . Then the ideal  $\bigoplus_{d \geq 1} (I_d \cap R_d)$  is a maximal element in  $\mathfrak{S}$ , as it is easily checked.

### 3. ELIMINATION THEORY

The main theorem of elimination theory may be formulated as follows. Let  $P_1, \dots, P_r$  be polynomials in  $k[X_0, X_1, \dots, X_n; Y_1, \dots, Y_m]$  with  $P_j$  homogeneous of degree  $d_j$  in the variables  $X_0, X_1, \dots, X_n$  alone, i.e. of the form

$$P_j = \sum_{\alpha_0 + \dots + \alpha_n = d_j} X_0^{\alpha_0} X_1^{\alpha_1} \dots X_n^{\alpha_n} f_{\alpha,j}(Y_1, \dots, Y_m)$$

where the  $f_{\alpha,j}$ 's are polynomials in  $k[Y_1, \dots, Y_m]$ .

Denote by  $J$  the ideal in  $k[X_0, X_1, \dots, X_n; Y_1, \dots, Y_m]$  generated by  $P_1, \dots, P_r$  and by  $\mathfrak{A}$  the ideal of polynomials  $f$  in  $k[Y_1, \dots, Y_m]$  with the following property (the so-called Hurwitz' Trägheitsformen):

(E) There exists an integer  $N \geq 1$  such that  $f X_0^N, f X_1^N, \dots, f X_n^N$  all belong to  $J$ .

As usual we denote by  $\mathbf{P}^n(K)$  the  $n$ -dimensional projective space over  $K$ .

**THEOREM C.** *Let  $V$  be the subset of  $\mathbf{P}^n(K) \times K^m$  consisting of the pairs  $(x, y)$  with  $x = (x_0 : x_1 : \dots : x_n)$  and  $y = (y_1, \dots, y_m)$  such that  $P_j(x_0, x_1, \dots, x_n; y_1, \dots, y_m) = 0$  for  $1 \leq j \leq r$ . Let  $W$  be the subset of  $K^m$  consisting of the vectors  $y$  such that  $Q(y) = 0$  for every  $Q$  in  $\mathfrak{A}$ . Then the projection of  $V \subset \mathbf{P}^n(K) \times K^m$  onto the second factor  $K^m$  is equal to  $W$ .*

To reformulate theorem C, let us consider the ring

$$B = k[X_0, X_1, \dots, X_n; Y_1, \dots, Y_m]$$

together with its subring  $B_0 = k[Y_1, \dots, Y_m]$ . Denote by  $B_d$  the  $B_0$ -module generated in  $B$  by the monomials of degree  $d$  in  $X_0, X_1, \dots, X_n$ . Then  $B = \bigoplus_{d \geq 0} B_d$  is a graded ring with  $J$  a graded ideal. Define the *graded ring*  $A = B/J$  with  $A_d = B_d/(B_d \cap J)$ . We have the following properties:

- (i) As a ring,  $A$  is generated by  $A_0 \cup A_1$ .
- (ii) For any nonnegative integer  $d$ ,  $A_d$  is a finitely generated module over  $A_0$ .

Furthermore, let  $\mathfrak{S}$  be the ideal in  $A_0$  consisting of all  $a$ 's such that  $aA_d = 0$  for all sufficiently large  $d$ 's, i.e. the union of the annihilators of the  $A_0$ -modules  $A_0, A_1, A_2, \dots$ .

**THEOREM D.** *Let  $A = \bigoplus_{d \geq 0} A_d$  be a graded commutative ring obeying hypotheses (i) and (ii) above. Let  $K$  be an algebraically closed field and  $\varphi : A_0 \rightarrow K$  be a ring homomorphism. In order that  $\varphi$  extend to a ring homomorphism  $\Psi : A \rightarrow K$  which does not annihilate the ideal  $A^+ = \bigoplus_{d \geq 1} A_d$  in  $A$ , it is necessary and sufficient that  $\varphi$  annihilate the ideal  $\mathfrak{S}$  defined above.*

We leave to the reader the simple proof of the necessity in theorem D as well as the derivation of theorem C from theorem D.

#### 4. PROOF OF THEOREM D

Let  $\mathfrak{P}$  be the kernel of  $\varphi$ , a prime ideal in  $A_0$ . Assume  $\mathfrak{S} \subset \mathfrak{P}$ . We subject the ring  $A$  to a number of transformations. At each step, the properties (i) and (ii) enunciated before the statement of theorem D will be preserved, as well as property  $A_d \neq 0$  for every  $d \geq 0$ . We shall mention what has been achieved after each step.

a) Factor  $A$  through the following graded ideal  $J$ : an element  $a$  in  $A$  belongs to  $J$  if and only if there exists an element  $s$  in  $A_0$  such  $s \notin \mathfrak{P}$  and  $sa = 0$ . For every  $d \geq 0$ , the annihilator  $\mathfrak{S}_d$  of the  $A_0$ -module  $A_d$  is contained in  $\mathfrak{S}$  hence in  $\mathfrak{P}$  and this implies  $J \cap A_d \neq A_d$ . Put  $A' = A/J$ ,  $\mathfrak{P}' = (\mathfrak{P} + J)/J$  and  $\Sigma = A'_0 - \mathfrak{P}'$ . Then any element in  $\Sigma$  is regular in  $A'$ .

b) Enlarge  $A'$  by replacing it by the subring  $A''$  of the total quotient ring of  $A'$  consisting of the fractions with denominators in  $\Sigma$ . Let  $A''_d$  be the set of fractions with numerator in  $A'_d$  and denominator in  $\Sigma$ ; then  $A'' = \bigoplus_{d \geq 0} A''_d$ . Then  $A''_0$  is a local ring with maximal ideal  $\mathfrak{P}'' = \mathfrak{P}' \cdot A'_0$ .

c) Factor  $A''$  through the graded ideal  $\mathfrak{P}'' \cdot A''$ . Since  $A''_d$  is a finitely generated module over the local ring  $A''_0$ , one gets  $A''_d \neq \mathfrak{P}'' A''_d$  by Nakayama's lemma. Put  $k = A''_0 / \mathfrak{P}''$ , and  $R = A'' / \mathfrak{P}'' A''$ .