CHAPTER IV PARTIAL DIFFERENTIAL EQUATIONS AND MEAN VALUE PROPERTIES

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **24 (1978)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **25.04.2024**

CHAPTER IV

PARTIAL DIFFERENTIAL EQUATIONS AND MEAN VALUE PROPERTIES

1. Invariant partial differential equations

We study in the present chapter a certain system of partial differential equations invariant under a finite reflection group G and related mean value properties. We assume throughout that the underlying field k is real (this permits us to introduce the methods of analysis) and that G is orthogonal, which can always be achieved after a linear change of variables. We rely on the invariant theory of the previous chapters to establish the forthcoming results. Conversely, we shall see that the problems studied in this chapter lead to a natural set of basic invariants for G. In the sequel, let R denote the ring of polynomials $k[x_1, ..., x_n]$. For any polynomial $p(x)$, $p(\partial)$ denotes the partial differential operator obtained by replacing $x = (x_1, ..., x_n)$ by the symbol

$$\partial = \partial_x = \left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\right).$$

We shall use the following result.

Theorem 4.1 (Fischer [9]). *Let $\mathfrak{a}$ be a homogeneous ideal of R (I.e. if $p \in \mathfrak{a}$, then each homogeneous block of $p \in \mathfrak{a}$). Let S be the space of polynomial solutions of $a(\partial)f = 0$, $a \in \mathfrak{a}$. Then $\mathfrak{a}$, S, R are vector spaces over k and $R = \mathfrak{a} \otimes S$.*

Proof. Let R_m = vector space of homogeneous polynomials of degree m, $0 \leqslant m < \infty$, $\mathfrak{a}_m = R_m \cap \mathfrak{a}$, $S_m = R_m \cap S$. We have $R = \sum_{m=0}^{\infty} \oplus R_m$, with similar expressions for $\mathfrak{a}$ and S. For any two polynomials P, Q, define $(P, Q) = P(\partial)\, Q\,|_{x=0}$. It is readily verified that (P, Q) is an inner product on R with $R_m \perp R_p$ whenever $m \neq p$. We show that $\mathfrak{a}_m$, S_m are orthogonal complements in R_m. Hence $R_m = \mathfrak{a}_m \oplus S_m$, $0 \leqslant m < \infty$, and so $R = \mathfrak{a} \oplus S$. $Q \in S$, $P \in \mathfrak{a}_m \Rightarrow P(\partial)\, Q(x) = 0 \Rightarrow (P, Q) = 0$. Hence $S_m \in \mathfrak{a}_m^{\perp}$. Let $Q \in \mathfrak{a}_m^{\perp}$. We show that $Q \in S_m$. It suffices to check that for any homogeneous $a \in \mathfrak{a}$ of degree $\leqslant m$, $a(\partial)\, Q(x) = 0 \Leftrightarrow b(\partial)\left[a(\partial)Q\right] = 0$ for all homogeneous b of degree $(m - \deg a)$. Now $b(\partial)\left[a(\partial)Q\right] = (ba, Q)$. Since $b\, a \in \mathfrak{a}_m$

and $Q \in \mathfrak{a}_m^{\perp}$, we conclude $b(\partial)[a(\partial)Q] = 0$. Thus $Q \in S_m$, so that $\mathfrak{a}_m^{\perp} \subset S_m$. It follows that $S_m = \mathfrak{a}_m^{\perp}$.

The following lemma will be required for the proof of Theorem 4.2.

LEMMA 4.1. Let $i(x)$ be an invariant of G and $\sigma \in G$. Let $f(x)$ be C^{∞} on an n-dimensional region $\mathscr{R}$. Then $i(\partial) f(\sigma x) = [i(\partial) f](\sigma x)$, provided $x, \sigma x \in \mathscr{R}$.

Proof. An application of the chain rule yields

$$i(\partial) f(\sigma x) = [i(\sigma^{-1}\partial)](\sigma x),$$

for any polynomial $i(x)$. If $i(x)$ is invariant under G, then $i(\sigma^{-1}x) = i(x)$, so that $i(\partial) f(\sigma x) = [i(\partial) f](\sigma x)$.

THEOREM 4.2. (Steinberg [21]). *Let* $\Pi(x) = \prod_{i=1}^{r} L_i(x)$, *where* $L_i(x) = 0$ *are the r.h.'s of* G, *and* $D\,\Pi$ = *linear span of partial derivatives of* $\Pi(x)$. *Let* S *be the solution space of* C^{∞} *functions on the* n-*dimensional region* $\mathscr{R}$ *satisfying* (4.1) $a(\partial) f = 0, x \in \mathscr{R}$ *and* $a \in \mathscr{I}$, $\mathscr{I}$ *being the ideal generated by all homogeneous invariants of* G *of positive degree. Then* $S = D\,\Pi$.

REMARK. If $O(n)$ is the orthogonal group acting on R^n, then it can easily be shown that $x_1^2 + \ldots x_n^2$ is a basis for the invariants of $O(n)$, i.e. each invariant polynomial is a polynomial in $x_1^2 + \ldots + x_n^2$. If we replace G by $O(n)$, then (4.1) reduces to Laplace's equation

$$\left(\frac{\partial^2}{\partial x_1^2} + \ldots + \frac{\partial^2}{\partial x_n^2}\right) f = 0 .$$

Because of this, it is natural to refer to the elements in S as the harmonic functions for G. Theorem 4.2 describes these harmonic functions.

Proof of Theorem 4.2. The inclusion $D\,\Pi \subset S$ clearly follows from $a(\partial)\,\Pi = 0$, $a \in \mathscr{I}$. It suffices to prove the latter for a homogeneous invariant of positive degree. By Lemma 3.4, $\Pi(\sigma x) = \det \sigma . \Pi(x)$, $\sigma \in G$. By Lemma 4.1, $[a(\partial)\Pi](\sigma x) = a(\partial)\,\Pi(\sigma x) = \det \sigma\,[a(\partial)\Pi]$. Thus $a(\partial)\,\Pi$ is skew. Again by Lemma 3.4, $\Pi \mid a(\partial)\,\Pi$. Since $\deg [a(\partial)\Pi] < \deg \Pi$, we must have $a(\partial)\,\Pi = 0$.

We now show that $S \subset D\,\Pi$. Let $f \in S$. We prove first that f is a polynomial x_i, $1 \leqslant i \leqslant n$, is a root of $P(X) = \prod_{\sigma \varepsilon G} [X - x_i(\sigma x)] = X^{|G|}$

$+ a_1 X^{|G|-1} + \ldots + a_{|G|}$, where the a_i's are homogeneous invariants of positive degree. Thus $x_i^{|G|} = -a_1 x_i^{|G|-1} \ldots a_{|G|} \in \mathcal{I}$, $1 \leqslant i \leqslant n$. The latter implies that every homogeneous polynomial $a(x)$ of degree $\geqslant n \mid G \mid$ is in $\mathcal{I}$. Hence $a(\partial) f = 0$, whenever $a(x)$ is homogeneous of degree $\geqslant n \mid G \mid \Rightarrow f$ is a polynomial of degree $< n \mid G \mid$. S is therefore a finite dimensional space of polynomials. In view of Fischer's Theorem $S \subset D\Pi \Leftrightarrow (D\Pi)^{\perp} \subset S^{\perp}$. A polynomial $P(x) \in (D\Pi)^{\perp} \Leftrightarrow (P, Q(\partial)\Pi) = 0 \ \forall$ polynomials $Q \Leftrightarrow Q(\partial)\,(P(\partial)\Pi)\,|_{x=0} \ \forall$ polynomials $Q \Leftrightarrow P(\partial)\,\Pi = 0$. We must therefore show that $P(\partial)\,\Pi = 0 \Rightarrow P \in \mathcal{I}$.

It suffices to prove this for homogeneous P. The result holds for $\deg P \geqslant n\,|G|$. Suppose that it holds for $\deg P = m + 1$. We show that it holds for $\deg P = m$ and, by induction, for arbitrary degree. Let $L(x) = 0$ be an r.h. of G. Then $L(\partial)\,P(\partial)\,\Pi(x) = 0$. By the induction hypothesis $L P \in \mathcal{I}$, so that

$$L(x)\,P(x) = \sum_{k=1}^{n} A_k(x)\,I_k(x) \tag{4.2}$$

where the A_k's are polynomials and $I_1, \ldots, I_n$ are a basic set of homogeneous invariants for G. Let σ be the reflection in the r.h. $L(x) = 0$. Substituting σx for x in (4.2) and subtracting the resulting equation from (4.1), we get

$$L(x)\,(P(x) + P(\sigma x)) = \sum_{k=1}^{n} (A_k(x) - A_k(\sigma x))\,I_k(x) \tag{4.3}$$

Each $[A_k(x) - A_k(\sigma x)] = 0$ whenever $L(x) = 0$. Thus

$$L(x) \mid [A_k(x) - A_k(\sigma x)]\,,$$

and

$$P(x) + P(\sigma x) = \sum_{k=1}^{n} \left[\frac{A_k(x) - A_k(\sigma x)}{L(x)}\right] I_k(x) \tag{4.4}$$

shows that $P(x) \equiv -P(\sigma x) \pmod{\mathcal{I}}$. Since the reflections in G generate G, we conclude from the latter that $P(x) \equiv \det\sigma\, P(\sigma x) \pmod{\mathcal{I}}$. Averaging over G, we obtain $P(x) \equiv P^*(x) \pmod{\mathcal{I}}$, where $P^*(x) = \dfrac{1}{|G|} \sum\limits_{\sigma \varepsilon G} \det\sigma \cdot P(\sigma x)$. We claim that $P^*(x)$ is skew. For if $\sigma_1 \in G$, then

$$\begin{aligned} P^*(\sigma_1 x) &= \frac{1}{|G|} \sum_{\sigma \varepsilon G} \det\sigma \cdot P(\sigma\sigma_1 x) \\ &= \frac{1}{\det\sigma_1} \sum_{\sigma \varepsilon G} \det\sigma\sigma_1\, P(\sigma\sigma_1 x) = \det\sigma_1\, P^*(x)\,. \end{aligned} \tag{4.5}$$

By lemma 3.4 $P^*(x) = \Pi(x)\, i(x)$, where i is a homogeneous invariant. If $\deg i > 0$, then $P^* \in \mathscr{I} \Rightarrow P \in \mathscr{I}$. Otherwise $P^* = c\,\Pi$, c a constant. By assumption $P(\partial)\,\Pi = 0$, while $a(\partial)\,\Pi = 0$ for $a \in \mathscr{I}$. It follows that $P^*(\partial)\,\Pi = c\,(\Pi, \Pi) \Rightarrow c = 0$, so that $P \equiv 0 \pmod{\mathscr{I}}$.

2. Mean Value Properties

We prove the equivalence of system (4.1) and a certain mean value property.

Theorem 4.3 (Steinberg [21]). *Let $f(x) \in C$ in the n-dimensional region $\mathscr{R}$ and let it satisfy the mean value property (m.v.p.)*

$$f(x) = \frac{1}{|G|} \sum_{\sigma \varepsilon G} f(x + \sigma y), \quad x \in \mathscr{R} \text{ and } \|y\| < \varepsilon_x, \tag{4.6}$$

where $\inf\limits_{x \in K} \varepsilon_x > 0$ for any compact subset K of $\mathscr{R}$ and $\|y\|^2 = \sum\limits_{i=1}^{n} y_i^2$. This m.v.p. is equivalent to having $f \in C^\infty$ and satisfying (4.1). It follows from Theorem 4.2 that the space S of continuous solutions to (4.6) $= D\,\Pi$.

Remark. The harmonic functions on $\mathscr{R}$ are characterized as the continuous functions on $\mathscr{R}$ satisfying the m.v.p. $f(x) = \int f(x+y)\, d\sigma(y)$, $x \in \mathscr{R}$ and $\|y\| < \varepsilon_x$, where $d\sigma(y)$ is the normalized Haar measure on the orthogonal group $O(n)$. (4.6) is just the G-analog of this m.v.p.

Proof of Theorem 4.3. Suppose first that $f(x)$ is C^∞ on $\mathscr{R}$ and satisfies (4.6). Let $a(x)$ be any homogeneous invariant of positive degree. Apply the operator $a(\partial_y)$ to both sides of (4.6). In view of Lemma 4.1, we get

$$\begin{aligned} 0 = a(\partial_y) f(x) &= \frac{1}{|G|} \sum_{\sigma \varepsilon G} a(\partial_y) f(x + \sigma y) \\ &= \frac{1}{|G|} \sum_{\sigma \varepsilon G} [a(\partial_y) f(x+y)](\sigma y) \end{aligned} \tag{4.7}$$

Use $a(\partial_y) f(x+y) = a(\partial_x) f(x+y)$ and set $y = 0$. We obtain $a(\partial_x) f(x) = 0$, $x \in \mathscr{R}$ and a any homogeneous invariant of positive degree. Hence $a(\partial_x) f(x) = 0$, $x \in \mathscr{R}$ and $a \in \mathscr{I}$. Since $\sum\limits_{i=1}^{n} x_i^2 \in \mathscr{I}$, we conclude in particular that $f(x)$ is harmonic on $\mathscr{R}$.

Suppose next that $f(x)$ is C on $\mathscr{R}$ and satisfies (4.6). Let $\{\delta_k\}$ be a sequence of C^∞ functions on R^n such that $\int \delta_k(x)\,dx = 1$, support of $\delta_k = \left\{x \mid \|x\| \leqslant \frac{1}{k}\right\}$, $\delta_k(x) \geqslant 0$ for all x and k. Let

$$f_k(x) = \int f(x-y)\,\delta_k(y)\,dy = \int f(y)\,\delta_k(x-y)\,dy\,.$$

It is readily checked that for any compact subset S of $\mathscr{R}$, $f_k(x) \in C^\infty$ on Int S (= interior of S) and satisfies (4.6) with $\mathscr{R}$ replaced by Int S, provided k is sufficiently large, and $f_k \to f$ uniformly on S as $k \to \infty$. For k sufficiently large, f_k is harmonic on Int S. It follows from Harnack's Theorem ([15], p. 248) that $f(x)$ is harmonic on $\mathscr{R}$. Hence $f(x)$ is real analytic on $\mathscr{R}$ ([15], p. 251) and so certainly C^∞ on $\mathscr{R}$.

Conversely let $f \in C^\infty$ on $\mathscr{R}$ and $a(\partial) f = 0$, $x \in \mathscr{R}$ and $a \in \mathscr{I}$. Then f is harmonic and so real analytic on $\mathscr{R}$. Hence there exists $\varepsilon_x > 0$ such that

$$f(x+y) = \sum_{m=0}^{\infty} \frac{1}{m!}(\partial_x, y)^m f(x),\ x \in \mathscr{R}$$

and $\|y\| < \varepsilon_x$. It follows that

$$\frac{1}{|G|}\sum_{\sigma \varepsilon G} f(x+\sigma y) = \sum_{m=0}^{\infty} \frac{P_m(\partial_x, y)}{m!} f(x),\ x \in \mathscr{R} \tag{4.8}$$

and $\|y\| < \varepsilon_x$ where

$$P_m(x, y) = \frac{1}{|G|}\sum_{\sigma \varepsilon G}(x, \sigma y)^m = \frac{1}{|G|}\sum_{\sigma \varepsilon G}(\sigma x, y)^m. \tag{4.9}$$

From (4.9), we see that for fixed y, each $P_m(x, y)$ is a homogeneous invariant polynomial in x of degree m. It follows that $P_m(\partial_x, y) f(x) = 0$, $x \in \mathscr{R}$ and $m \leqslant 1$, and (4.8) reduces to (4.6).

The solution space to either (4.1) or (4.6) is the finite dimensional vector space $D\,\Pi$. The following result gives further information on $D\,\Pi$.

THEOREM 4.4 (Chevalley [4]). *Let* S_m = *vector space of homogeneous polynomials of degree* m *in* $D\,\Pi$, $0 \leqslant m < \infty$, *so that* $D\,\Pi = \sum_{m=0}^{\infty} \oplus S_m$. *Let* $d_1, \ldots, d_n$ *be the degrees of the basic homogeneous invariants for* G. *Then*

$$\sum_{m=0}^{\infty} (\dim S_m)\, t^m = \prod_{i=1}^{n} \frac{1 - t^{d_i}}{1 - t} \tag{4.10}$$

and $\dim D\Pi = |G|$.

We prove first the preliminary

LEMMA 4.2. Let $R = k[x_1, ..., x_n]$ = ring of polynomials in $x_1, ..., x_n$ with coefficients from k, k being any field of characteristic 0. Let G be a finite reflection group acting on k^n and $\mathscr{I}$ the ideal generated by homogeneous invariants of positive degree. For any polynomial P, let $\bar{P}$ be its residue class in the residue class ring $R/\mathscr{I}$. Suppose that $P_1, ..., P_s$ are homogeneous polynomials such that $\bar{P}_1, ..., \bar{P}_s$ are linearly independent over $R/\mathscr{I}$ (the latter is a vector space over k). Then $P_1, ..., P_s$ are linearly independent over $k(I)$, the field obtained by adjoining the set I of all invariant polynomials to k.

Proof. Suppose $\sum_{i=1}^{s} V_i P_i = 0$ where $V_i \in k(I)$, $1 \leqslant i \leqslant s$. We may suppose that the V_i's are homogeneous and [$\deg V_i + \deg P_i$] is the same for all i. Let $I_1, ..., I_n$ be a basic set of homogeneous invariants of positive degree. Let S_j, $0 \leqslant j < \infty$, be the different monomials in $I_1 \ldots I_n$ arranged by increasing x-degree, with $s_0 = 1$. Let $V_i = \sum_{j=0}^{\infty} k_{ij} S_j$, $1 \leqslant i \leqslant s$, the k_{ij}'s being elements of k, and define k_{i0} to be 0. We have

$$\sum_{i=1}^{s} V_i P_i = \sum_{j=0}^{\infty} \left[\sum_{i=1}^{s} k_{ij} P_i \right] S_j = 0 \tag{4.11}$$

Assume, as induction hypothesis, that $k_{ij} = 0$ for $j < l$. Thus $\sum_{j=l}^{\infty} [\sum_{i=1}^{s} k_{ij} P_i] S_j = 0$. $S_l \notin$ ideal generated by the S_j's, $j > l$, as $I_1, ..., I_n$ are algebraically independent. It follows from Lemma 2.1 that $\sum_{i=1}^{s} k_{il} P_i \in \mathscr{I} \Leftrightarrow \sum_{i=1}^{s} k_{il} \bar{P}_i = 0 \Leftrightarrow k_{il} = 0$, $1 \leqslant i \leqslant s$. Hence all $k_{ij} = 0$ and $V_i = 0$, $1 \leqslant i \leqslant s$. I.e. $P_1, ..., P_s$ are linearly independent over $k(I)$.

We now return to the proof of Theorem 4.4. Let $A_1, ..., A_q$ be homogeneous polynomials such that $\bar{A}_1, ..., \bar{A}_q$ form a basis for $R/\mathscr{I}$. By induction on the degree, we see that every polynomial P may be expressed as

$$P = \sum_{i=1}^{q} J_i A_i \tag{4.12}$$

where the J_i's are invariant polynomials. Lemma 4.2 shows that this representation is unique. Let R_m = set of homogeneous polynomials of degree m, $I_m = I \cap R_m$, $(R/\mathscr{I})_m$ = vector space spanned by those $\bar{A}_i'$s for which degree $A_i = m$. Let

$$\mathfrak{p}_R(t) = \sum_{n=0}^{\infty} (\dim R_m)\, t^m, \quad \mathfrak{p}_I(t) = \sum_{m=0}^{\infty} (\dim I_m)\, t^m,$$

$$\mathfrak{p}_{R\mathscr{I}}(t) = \sum_{m=0}^{\infty} \dim (R/\mathscr{I})_m\, t^m.$$

In view of the uniqueness of the representation (4.12), we have

$$\mathfrak{p}_R(t) = \mathfrak{p}_I(t)\, \mathfrak{p}_{R/\mathscr{I}}(t) \tag{4.13}$$

Now

$$\mathfrak{p}_I(t) = \frac{1}{\prod_{i=1}^{n} (1 - t^{d_i})} \qquad \text{(formula (2.5))}$$

while

$$\mathfrak{p}_R(t) = \frac{1}{(1-t)^n}$$

(as $\dim R_m = \binom{m+n-1}{m}$). By Fischer's Theorem $R/\mathscr{I}$ may be identified with $D\,\Pi$, so that $\mathfrak{p}_{R/\mathscr{I}}(t) = \sum_{m=0}^{\infty} (\dim S_m)\, t^m$. Thus (4.13) becomes (4.10).

Set $t = 1$ in (4.10). The left side becomes $\sum_{m=0}^{\infty} \dim S_m = \dim D\,\Pi$. Since

$$\frac{1 - t^{d_i}}{1 - t} = 1 + t + \ldots + t^{d_i - 1} = d_i$$

at $t = 1$, the right side becomes $\prod_{i=1}^{n} d_i = |G|$ (by Theorem 2.2). Thus $\dim D\,\Pi = |G|$.

We now describe the solution space to (4.6) when we restrict the direction of y. For simplicity, we restrict ourselves to irreducible groups (the reducible case is discussed in [12]).

THEOREM 4.5. *Let $f(x) \in C$ in the n-dimensional region $\mathscr{R}$ and satisfy the m.v.p.*

$$f(x) = \frac{1}{|G|} \sum_{\sigma \varepsilon G} f(x + t\sigma y), \; x \in \mathscr{R} \text{ and } 0 < t < \varepsilon_x, \tag{4.14}$$

$\inf_{x \in K} \varepsilon_x > 0$ for any compact subset K of $\mathscr{R}$ and y denoting a fixed vector $\neq 0$. This m.v.p. is equivalent to having $f \in C^\infty$ on $\mathscr{R}$ and $P_m(\partial_x, y) f = 0$, $x \in \mathscr{R}$ and $1 \leqslant m < \infty$, P_m being defined by (4.9).

Proof. Suppose first that $f \in C^\infty$ on $\mathscr{R}$ and satisfies (4.14). Using the finite Taylor expansion for $f(x + t\sigma y)$, we get for each integer $N \geqslant 0$

$$(4.15) \qquad 0 = \sum_{m=1}^{N} \left[\frac{P_m(\partial_x, y) f}{m\,!} \right] t^m + O(t^{N+1}) \text{ as } t \to 0\,.$$

Dividing by successive powers of t and letting $t \to 0$, we conclude $P_m(\partial_x, y) f = 0$, $x \in \mathscr{R}$ and $1 \leqslant m < \infty$. If $f \in C$, then we argue as in the proof of Theorem 4.3, introducing the functions f_k. For any compact subset S of $\mathscr{R}$ and k sufficiently large, the f_k's will be C^∞ on Int S and satisfy there $P_m(\partial_x, y) f = 0$, $1 \leqslant m < \infty$. $P_2(x, y)$ is a non-zero homogeneous invariant of degree 2. For irreducible G, there is up to a multiplicative constant, only one such invariant, namely $\sum_{i=1}^{n} x_i^2$. Thus $P_2(x, y) = c(y) \sum_{i=1}^{n} x_i^2$, where $c(y) \neq 0$ is a constant depending on y. Thus for k sufficiently large, $f_k(x)$ is harmonic on Int S. Since $f_k \to f$ uniformly on compact subsets of $\mathscr{R}$, $f(x)$ is harmonic on $\mathscr{R}$ and hence certainly C^∞ on $\mathscr{R}$.

Conversely, let $P_m(\partial_x, y) f = 0$, $x \in \mathscr{R}$ and $1 \leqslant m < \infty$. Since $P_2(\partial_x, y) f = 0$, f is harmonic and so real analytic on $\mathscr{R}$. It follows that there exists $\varepsilon_x > 0$ such that

$$(4.16) \qquad \frac{1}{|G|} \sum_{\sigma \varepsilon G} f(x + t\sigma y) = \sum_{m=0}^{\infty} \left[\frac{P_m(\partial_x, y) f}{m\,!} \right] t^m, \; x \in \mathscr{R}$$

and $0 < t < \varepsilon_x$.

Since $P_m(\partial_x, y) f = 0$, $x \in \mathscr{R}$ and $1 \leqslant m < \infty$, (4.16) reduces to (4.14).

We shall describe the solution space to $P_m(\partial_x, y) f = 0, 1 \leqslant m < \infty$, y being a fixed vector $\neq 0$. We first prove some preliminary lemmas.

Lemma 4.3. *Let $\mathscr{C}$ be a collection of homogeneous polynomials in $k[x_1 \ldots, x_n]$ of positive degree, k being a field of characteristic 0. Let G be a finite reflection group acting on k^n. The following conditions are equivalent.*

i) *$\mathscr{C}$ is a basis for the invariants of G*

ii) $\mathscr{C}$ is a basis for the ideal $\mathscr{I}$ generated by the homogeneous invariants of positive degree.

iii) Let $d_1, ..., d_n$ be the degrees of the basic homogeneous invariants of G.

For each d_i there exists a polynomial $P_i \in \mathscr{C}$ of degree d_i such that

$$\frac{\partial (P_1, \ldots, P_n)}{\partial (x_1, \ldots, x_n)} \neq 0 .$$

Proof. Let $\mathscr{I}(\mathscr{C})$ = ideal generated by $\mathscr{C}$, so that $\mathscr{I}(\mathscr{C}) \subset \mathscr{I}$. If i) holds, then $\mathscr{I}(\mathscr{C})$ contains every homogeneous invariant of positive degree, so that $\mathscr{I} \subset \mathscr{I}(\mathscr{C}) \Rightarrow \mathscr{I} = \mathscr{I}(\mathscr{C})$.
Thus i) $\Rightarrow$ ii).

Suppose ii) holds. Choose in $\mathscr{C}$ a minimal basis for $\mathscr{I}$. The proof of Chevalley's Theorem shows that this minimal basis consists of n homogeneous invariants $P_1, ..., P_n$ which are algebraically independent

$$\Leftrightarrow \frac{\partial (P_1, \ldots, P_n)}{\partial (x_1, \ldots, x_n)} \neq 0 .$$

According to Theorem 3.1, these degrees must be $d_1, ..., d_n$. Thus ii) $\Rightarrow$ iii).

Finally, the implication iii) $\Rightarrow$ i) is contained in Theorem 3.13.

LEMMA 4.4. Let G be a finite reflection group acting on k^n. Let $I_1, ..., I_n$ be a basic set of homogeneous invariants of respective positive degrees $d_1, ..., d_n$ which are assumed distinct; i.e. $d_1 < d_2 < ... < d_n$. Let $P_1, ..., P_n$ be another set of homogeneous invariants of respective degrees $d_1, ..., d_n$. Thus

$$\begin{aligned} P_i(x) &= F_i(I_1(x), \ldots, I_{i-1}(x)) + c_i I_i(x) \\ &= F_i(x) + c_i I_i(x), \; 1 \leqslant i \leqslant n \end{aligned} \tag{4.17}$$

where $F_i(x)$ is homogeneous of degree m_i, with $F_1 = 0$, and c_i a constant. Then

$$\frac{\partial (P_1, \ldots, P_n)}{\partial (x_1, \ldots, x_n)} = c_1 \ldots c_n \frac{\partial (I_1, \ldots, I_n)}{\partial (x_1, \ldots, x_n)} \tag{4.18}$$

Proof. We have

$$\frac{\partial (P_1, \ldots, P_n)}{\partial (x_1, \ldots, x_n)} = \frac{\partial (F_1, \ldots, F_n)}{\partial (I_1, \ldots, I_n)} \frac{\partial (I_1, \ldots, I_n)}{\partial (x_1, \ldots, x_n)}$$

The matrix $\left[\dfrac{\partial F_i}{\partial I_j}\right]$ is triangular and $\dfrac{\partial F_i}{\partial I_i} = c_i$, $1 \leqslant i \leqslant n$, so that

$$\frac{\partial (F_1, \ldots, F_n)}{\partial (x_1, \ldots, x_n)} = c_1 \ldots c_n.$$

THEOREM 4.6 (Flatto and Wiener [10]). i) *Let* S_y *be space of continuous functions on the* n*-dimensional region* $\mathscr{R}$ *satisfying the mean value property (4.14).* $S_y = D\,\Pi$ *iff* $G \neq D_{2n}$, $2 \leqslant n < \infty$, *and*

$$\frac{\partial (P_{d_1}, \ldots, P_{d_n})}{\partial (x_1, \ldots, x_n)} \neq 0 .$$

ii) *For* $G \neq D_{2n}$, $2 \leqslant n < \infty$, *we have*

$$\frac{\partial (P_{d_1}, \ldots, P_{d_n})}{\partial (x_1, \ldots, x_n)} = J_1(y) \ldots J_n(y)\, \Pi(x) \tag{4.19}$$

the J*'s being a basic set of homogeneous invariants for* G. *Hence*

$$S_y = D\,\Pi \text{ iff } J_1(y) \ldots J_n(y) \neq 0 .$$

Proof. According to Theorem 4.5, S is the solution space of

$$f \in C^\infty \text{ and } p(\partial) f = 0,\ x \in \mathscr{R} \text{ and } p \in \mathscr{P}_y. \tag{4.20}$$

where $\mathscr{P}_y = (P_1(x, y), \ldots, P_m(x, y), \ldots)$. It follows from Theorems 4.1, 4.2 that $S_y = D\,\Pi$ iff $\mathscr{P}_y = \mathscr{I}$. By Lemma 4.3, $\mathscr{P}_y = \mathscr{I}$ iff the degrees $d_1, \ldots, d_n$ are distinct and

$$\frac{\partial (P_{d_1}, \ldots, P_{d_n})}{\partial (x_1, \ldots, x_n)} \neq 0$$

An inspection of the table in section 3.3 reveals that the d_i's are distinct except when $G = D_{2n}$, $2 \leqslant n < \infty$, in which case two d_i's equal $2n$. ii) For each n-tuple $a = (a_1, \ldots, a_n)$ of non-negative integers, let $J_a(x) = \dfrac{1}{|G|} \sum_{\sigma \in G} (\sigma x)^a$. We have

$$P_m(x, y) = \frac{1}{|G|} \sum_{\sigma \varepsilon G} (\sigma x, y)^m = \frac{1}{|G|^2} \sum_{\sigma_1 \varepsilon G} \sum_{\sigma_2 \varepsilon G} (\sigma_1 x, \sigma_2 y)^m =$$

$$\frac{1}{|G|^2} \sum_{|a| = m} \sum_{\sigma_1 \varepsilon G} \sum_{\sigma_2 \varepsilon G} \frac{m!}{a!} (\sigma_1 x)^a (\sigma_2 y)^a = \sum_{|a| = m} \frac{m!}{a!} J_a(x)\, J_a(y) \tag{4.21}$$

Let $I_1, \ldots, I_n$ be a basic set of homogeneous invariants of respective degrees $d_1, \ldots, d_n$. Let $|a| = d_i$, $1 \leqslant i \leqslant n$. Then

$$(4.22)\qquad J_a(x) = F_a\big(I_1(x), \ldots, I_{i-1}(x)\big) + c_a I_i(x) = F_a(x) + c_a I_i(x)$$

where $F_a(x)$ is homogeneous of degree d_i with $F_a(x) = 0$ for $i = 1$, and c_a is a constant. (4.21), (4.22) give

$$(4.23)\qquad P_{d_i}(x, y) = \sum_{|a|=d_i} \frac{d_i!}{a!} J_a(y) F_a(x) + J_i(y) I_i(x),\ 1 \leqslant i \leqslant n$$

where

$$(4.24)\qquad J_i(y) = \sum_{|a|=d_i} \frac{d_i!}{a!} c_a J_a(y),\ 1 \leqslant i \leqslant n$$

(4.19) follows from (4.23) and Lemma 4.4. J_i is homogeneous of degree d_i. We show that $J_1, \ldots, J_n$ are algebraically independent and thus conclude from Lemma 4.3 that $J_1, \ldots, J_n$ form a basis for the invariants of G. Now the J_a's form a basis for the invariants of G (see Noether's proof of Theorem 1.1). Hence, by Lemma 4.3, there exists $n\, J_a'$s of respective degrees $d_1, \ldots, d_n$ which are algebraically independent. By Lemma 4.4, for each of these J_a's, $c_a \neq 0$. (4.22), (4.24) give

$$(4.25)\quad J_i(y) = \sum_{|a|=d_i} \frac{d_i!}{a!} c_a F_a(y) + \Big(\sum_{|a|=m_i} \frac{d_i}{a!} c_a^2\Big) I_i(y),\ 1 \leqslant i \leqslant n$$

For each $1 \leqslant i \leqslant n$, there exists an a such that $|a| = d_i$ and $c_a \neq 0$, so that the n constants $\sum_{|n|=d_i} \frac{d_i}{a!} c_a^2$ are all $\neq 0$. It follows from (4.25) and Lemma 4.4, that $J_1, \ldots, J_n$ are algebraically independent.

The following theorem yields an algebraic characterization of the J_i's.

THEOREM 4.7 [12]. $J_1(x) = c \sum_{i=1}^{n} x_i^2$, $c \neq 0$. *For* $2 \leqslant i \leqslant n$, $J_i(x)$ *is determined up to a constant as the homogeneous invariant of degree* d_i *which satisfies the differential equations* $J_k(\partial) J_i(x) = 0$, $1 \leqslant k < i$.

Proof. $J_1(x)$ is a non-zero homogeneous invariant of degree 2 and must therefore be a non-zero multiple of $\sum_{i=1}^{n} x_i^2$. Let $2 \leqslant i \leqslant n$ and $1 \leqslant k < d_i$. Let $Q(x)$ be an arbitrary homogeneous invariant polynomial of degree k. We have

$$(4.26)\qquad Q(\partial_y) P_m(x, y) = Q(\partial_y) \Big[\frac{1}{|G|} \sum_{\sigma \varepsilon G} (y, \sigma x)^m\Big]$$

$$= m(m-1)\ldots(m-k+1) P_{m-k}(x, y)\ Q(x)$$

From (4.23), we obtain

$$(4.27)\qquad Q(\partial_y) P_{d_i}(x, y) = \sum_{|a|=d_i} \frac{d_i!}{a!} [Q(\partial) J_a(y)] F_a(x) + [Q(\partial) J_i(y)] I_i(x), \quad 1 \leqslant i \leqslant n$$

so that

$$(4.28)\qquad d_i(d_i-1) \cdots (d_i-k+1) P_{d_i-k}(x, y) Q(x) = \sum_{|a|=d_i} \frac{d_i!}{a!} [Q(\partial) J_a(y)] F_a(x) + [Q(\partial) J_i(y)] I_i(x), \quad 1 \leqslant i \leqslant n$$

Suppose that $Q(\partial) J_i(y) \neq 0$. Choose y_0 so that $Q(\partial) J_i(y) \neq 0$ at y_0. Let $y = y_0$ in (4.28). The polynomial $P_{d_i-k}(x, y_0)$ has degree $< d_i$ and thus is a polynomial in $I_1, (x), \ldots, I_{i-1}(x)$. Each F_a is also a polynomial in $I_1, \ldots, I_{i-1}$. We conclude from (4.28) that $I_1, \ldots, I_i$ are algebraically dependent, a contradiction. Hence $Q(\partial) J_k(y) = 0$, so that $J_k(\partial) J_i(x) = 0, 1 \leqslant k < i$.

The conditions of Theorem 4.7 determine J_i up to a constant. For let V_i = space of homogeneous invariants of degree d_i, W_i = space of homogeneous invariants of degree d_i spanned by the monomials in $I_1, \ldots, I_{i-1}$. Then $\dim V_i = \dim W + 1$. For any $J \in V_i$, the conditions $J_k(\partial) J(x) = 0, 1 \leqslant k < i$, are equivalent to $J \in W_i^{\perp}$. Since $\dim W_i^{\perp} = \dim V_i - \dim W_i = 1$, we conclude that J_i is determined up to a constant.

COROLLARY. The manifold $\mathcal{M} = \{y \mid J_1(y) \cdots J_n(y) = 0\}$ contains real points $y \neq 0$. I.e. there exists $y \in R^n$ such that $S \neq D\,\Pi$.

Proof. For $2 \leqslant i \leqslant n$, $J_1(\partial) J_i(x) = 0$. Since $J_1(x) = c \sum_{i=1}^{n} x_i^2$, $c \neq 0$, this means that $J_i(x)$ is harmonic. By the mean value property for harmonic functions, the average value of $J_i(y)$ on a sphere of radius $r > 0 = J_i(0) = 0$. Thus $J_i(y)$ must change sign on this sphere and a connectedness argument yields the existence of a $y \neq 0$ for which $J_i(y) = 0$.

In view of Theorem 4.6, we call $\mathcal{M}$ the "exceptional manifold" for G and the non-zero vectors y of $\mathcal{M}$, the "exceptional directions" for G. A geometric description of $\mathcal{M}$ is given in [24] for the groups H_2^n and A_3. There remains the problem of describing the solution space S_y to the m.v.p. (4.14) in case y is an exceptional direction, as $D\,\Pi$ is then a proper subspace of S_y. This seems to be a difficult problem. In [11], it is solved for the groups H_2^n, A_3.