ON THE CHARACTERISTIC CLASSES OF GROUPS OF DIFFEOMORPHISMS [1]

by Raoul BOTT [2]

It gives me great pleasure to join in this celebration of Beno Eckmann's sixtieth birthday. I had the good fortune to meet Beno at the very outset of my mathematical career in 1950 at Princeton. Of course at that time I knew much too little of the subject to have other than neighbourly conversations with him—they lived next to us in the housing project—but over the years I have, like all of us, learned much from his superb lectures, expositions and clarifying point of view. I have always been especially fond of his paper on the Radon Hurwitz numbers, and it played an important part in Shapiro and my thinking about Clifford Algebras.

But let me turn to my talk, I am not allowed to reminisce indefinitely. The question I would like to discuss here has fascinated me for the past six or seven years and is related to the work of Gelfand Fuks on the one hand, and the work of Haefliger on Foliation and that of Chern, Simons and Cheeger on the other. Here I will however launch into the subject by presenting it from the point of view of group cohomology, which I take to be the common denominator of all of us here.

Consider then an abstract group Γ acting (on the right) in C^∞ manner on the smooth manifold M:

$$(1) \qquad M \times \Gamma \to M$$

If one the chooses a volume θ for M, then every $f \in \Gamma$ maps θ into a multiple of itself:

$$f^* \theta = \mu(f)\, \theta,$$

and out of this magnification factor we may manufacture a function of $(n+1)$ arguments, $\{f_1, \ldots, f_{n+1}\{$ in Γ, $n = \dim M$, by setting

$$(2) \qquad \omega(f_1, \ldots, f_{n+1}) = \int_M log\, \mu_1(\bar{f}_1) \cdot d\, log\, \mu(\bar{f}_2) \ldots d\, log\, \mu(\bar{f}_{n+1}),$$

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where

$$\bar{f}_i = f_1 \circ f_2 \circ \ldots \circ f_i .$$

More generally let us choose a Riemann structure on M, and let ∇ denote the associated Levicivita connection on M.

Then $f \in \Gamma$ acts on ∇ and the difference

$$f^* \nabla - \nabla = \xi(f)$$

is a well-defined 1-form on M with values in the bundle of endomorphisms of T, the tangent bundle of M:

$$\xi(f) \in \Omega^1(M; \mathrm{End}\, T) .$$

Then for any partition $\alpha_1 + \ldots + \alpha_k = n$ of n we define functions $\omega^\alpha(f_1, \ldots, f_{n+1})$ by the recepee:

$$(3) \quad \omega^\alpha(f_1, f_2 \ldots f_{n+1}) = \int_M \Sigma \, log \, \mu_1 \, \mathrm{Trace}\,(\xi_2 \ldots \xi_{\alpha_1}) \ldots \mathrm{Trace}\,(\xi_{\alpha_k + 1}, \ldots \xi_{\alpha_{n+1}}) ,$$

where μ_i stands for $\mu(\bar{f}_i)$ computed for the volume of the Riemann structure, and ξ_i stands for $\xi(\bar{f}_i)$. Finally the sum is taken alternatingly.

Concerning these expressions one now has the following

PROPOSITION. *The* $(n+1)$*-cochains* ω *and* ω^α *just defined are cocycles and their classes*

$$(4) \qquad [\omega^\alpha] \in H^{n+1}(\Gamma, \mathbf{R})$$

are independent of the choices involved.

By the way the formula (1.1) for ω, which goes back to Thurston, agrees with $\omega^{1, \cdots 1}$, up to a constant, because if θ is taken as volume of the Riemannian structure, then one easily finds that

$$\mathrm{trace}\, \xi(f) = -d \, log \, \mu(f) .$$

The classes ω^α are therefore certain invariants of the action of Γ on M, and I propose to call them characteristic numbers because they are constructed in analogy with the usual ones.

Several questions arise now naturally. First of all, where did these formulas come from. Next, how many similar formulas could one expect, and finally are these classes independent in the sense that for any linear non-trivial combination

$$\psi = \Sigma a_\alpha \omega^\alpha ,$$

there is some action of some Γ on some M so that ψ is nonzero in $H^{n+1}(\Gamma)$.

The first question we can answer quite precisely. The second one really only when $M = S^1$, although the work that André Haefliger is reporting on reduces the problem to a purely computational one whenever the action of the group of Diffeomorphisms on the principal bundle of M is known. Thus for surfaces and in view of the recent result of A. Hatcher that Diff $(S^3) \sim O(4)$ also for the three sphere, this question is essentially answered. On the other hand the third question is not even decided for S^1.

Here I will address myself mainly to the first one and will also try to indicate my personal recepee for deriving the explicit cocycles I have written down. The first step in this direction is then to explain that the classes ω^α have their antecedents in the characteristic classes of foliations, and appear in $H^*(\Gamma)$ by virtue of the fact that the associated bundle

$$\begin{array}{c} M\Gamma \\ \downarrow \pi \\ *\Gamma \end{array} \tag{5}$$

over the classifying space of Γ given by the action of Γ, admits a canonical foliation $F\Gamma$, transversal to the fiber.

Let me start therefore with a quick reveiw of the characteristic classes of a Foliation. There are really two quite different ways of doing this, sort of a high and a low road to the same goal. The high road is closer to Gelfand Fuks theory. while the low road is closer to the considerations of Chern, Simons, Cheeger. For our purposes this is the best way to start.

Recall first that a foliation F on M is defined by a sub-bundle $E \in T$ whose sections $S(E)$ are closed under the lie bracket:

$$[S(E), S(E)[\in S(E) . \tag{6}$$

I often denote a foliation by the associated exact sequences

$$F: 0 \to E \to T \xrightarrow{\pi} Q \to 0$$

induced by the inclusion of E in T.

Now in the usual theory of characteristic classes every connection ∇ on Q, induces a well defined homomorphism

$$\varphi(\nabla)^*: \mathbf{R}[c_1, \ldots, c_q] \to \Omega^*(M)$$

of the polynomial ring in the universal Chern-classes, c_i in dim $2i$, to the forms on M. Quite a long time ago I noticed that for a special class of connections on Q—called the *basic* ones—this homomorphism factors through the quotient ring

$$(7) \qquad \underline{\mathbf{R}}[c_1, \ldots, c_q] = \mathbf{R}[c_1, \ldots, c_q] / (\dim > 2q)$$

in which the ideal of classes in dim $> 2q$ has been set equal to zero. These basic connections are characterised by the condition that,

$$(8) \qquad \Delta_X^B \cdot \tilde{Y} = \pi[X, \tilde{Y}], \text{ for } X \in S(E),\ Y \in S(Q), \quad \tilde{Y} \in S(T),\ \pi \tilde{Y} = Y,$$

and the integrability enters here, because the right hand side is independent of the choice of $\tilde{Y}$ if and only if E is integrable.

In any case then, from such a ∇^B, we have the commutative diagram:

$$(9) \qquad \begin{array}{ccc} \mathbf{R}[c_1 \ldots c_q] & \longrightarrow & \Omega^*(M) \\ \downarrow & \nearrow & \\ \underline{\mathbf{R}}[c_1 \ldots c_q] & & \end{array}$$

expressing a vanishing phenomenon which yields the only known topological obstruction to integrability.

Now it is hardly new in topological circles that if something vanishes then something else should simultaneously appear somewhere. Nevertheless it took many years and the work of Godbillon-Vey-Roussarie, and for them the work of Gelfand Fuks was the inspiration, as well as ideas of Chern-Simons, conversation with Milnor, letters with A. Haefliger until we discovered the classes which "appear" as a result of (9).

The answer is as follows. One first constructs a complex

$$WO_q = \underline{\mathbf{R}}[c_1, \ldots, c_q] \otimes E(h_1, h_3, \ldots, h_p), \ p \text{ odd} \leqslant q$$

with

$$d\, h_i = c_i, \ i \text{ odd}$$

and hence analogous to the complex introduced in the Gelfand Fuks work, describing $H^*(a_n)$ the cohomology of formal vector-fields in $\mathbf{R}^n$. Then in terms of this complex the correct extension of (9) yields a *cannonical homotopy class of maps*:

$$(10) \qquad \varphi^*(F): WO_q \to \Omega^*(M)$$

for any foliation F.

A representative of $\varphi^*(F)$ is obtained by choosing two connections on Q, namely a basic one ∇^B and a Riemannian one ∇^R, that is some connection preserving some Riemann structure on Q. Indeed the standard comparison procedure between connections then yields difference forms,

$$h_i(\nabla^B, \nabla^R) \in \Omega^*(M)$$

with

$$c_i(\nabla^B) - c_i(\nabla^R) = d\, h_i(\nabla^B, \nabla^R),$$

and $\varphi^*(F)$ is then induced by the map

$$\begin{aligned} c_i &\to c_i(\nabla^B) \\ h_i &\to h_i(\nabla^B, \nabla^R) \quad i \text{ odd}. \end{aligned}$$

Because $c_i(\nabla^R) \equiv 0$ for odd i, in view of the special symmetries of a Riemannian connection, this $\varphi^*(F)$ is then well defined. Finally the homotopy class of $\varphi^*(F)$ is seen to be independent of all choices because both the basic and the Riemann connections form convex sets.

The cohomology of the algebras WO_q can be computed explicitly without too much trouble. For instance for $n = 1$, H^* has classes only in dim 1 and 3:

$$H^*(WO_1) = \{1; h_1 c_1\}$$

while

$$H^*(WO_2) = \{1; c_2; h_1 c_1^2, h_1 c_2\}.$$

In general a basis for $H^*(WO_n)$ in terms of monomials $h_\alpha \cdot c_\beta$ $h_{\alpha_1} \dots h_{\alpha_k} c_{\beta_1} \dots c_{\beta_n}$. with the α, β subject to certain inequalities is known, and all "exotic classes" i.e. ones involving an h, are annihilated by multiplication with any element of positive dimension. The c_{2i} of course correspond to the usual Pontryagin classes.

Finally note that there is a map

$$WO_{q+1} \to WO_q$$

defined by sending c_i to c_i and h_i to h_i for $i \leqslant q$, and sending c_{q+1} to 0 as well as h_{q+1} — if $(q+1)$ is odd. The image of *this map constitutes the stable classes* of WO_q. The simplest of these are of course the Pontryagin classes of Q.

So much then for a quick review of the characteristic classes of foliations. As mentioned before, they are pertinant to our discussion because the bundle

$$\begin{array}{c} M\Gamma \\ \pi \downarrow \\ *\Gamma \end{array}$$

has a canonical foliation $F\Gamma$, transversal to the fibers M, of π. Hence in particular the classes $h_1 c^\alpha \in H^{2n+1}(WO_n)$, $n = \dim M$, can be applied to $F\Gamma$. Finally integrating over the fiber in π, we obtain classes $\pi_* h_1 c^\alpha (F\Gamma)$ in $H^{n+1}(*\Gamma) = H^{n+1}(\Gamma)$ and I claim that the formulae I wrote down at the start, are explicit recepees for these classes. Precisely we have:

Let the s_i and c_i be related as the power sums are to the elementary functions, that is by the Newton formulae. Thus

$$\begin{aligned} s_1 - c_1 &= 0 \\ s_2 - s_1 c_1 + 2c_2 &= 0 \\ s_n - s_{n-1} c_1 + \dots \pm n c_n &= 0, \end{aligned}$$

and let $s^\alpha = s_1^{\alpha 1} \dots s_n^{\alpha n}$. *Then, the form* $\omega^\alpha (f_1 \dots f_{n+1})$ *represents a constant multiple of* $\pi_* h_1 s^\alpha (F\Gamma)$:

(11) $$[\omega^\alpha] \cong \pi_* h_1 s^\alpha (F\Gamma).$$

To prove such a result in any particular case one could in principle start with any free action of Γ on a manifold E which is highly connected so that E/Γ is a high approximation to the classifying space $B\Gamma \equiv *\Gamma$, of Γ, and then compute our classes in

$$\begin{gathered} E \times M \\ \Gamma \end{gathered}$$

with $F(\Gamma)$ being the foliation, which, when lifted to $E \times M$, is given by the "horizontal space" TE of tangents along E:

$$\pi^{-1}(F\Gamma) = TE \subset T(E \times M).$$

However for explicit formulas which are valid in general it is best to extend the concepts of *foliation* of *de Rham theory*, and of *characteristic classes* to a larger category in which explicit models of the classifying space $B\Gamma$ and its associated bundle $M \cdot \Gamma$ can be constructed.

The appropriate category for such a program is the category of *simplicial manifolds* where such explicit models are by now well known. Thus $M\Gamma$ can be represented by:

(12) $$M\Gamma \colon M \leftleftarrows M \times \Gamma \,\substack{\leftarrow \\ \leftarrow \\ \leftarrow}\, M \times \Gamma \times \Gamma \dots,$$

with the bundary maps given by

$$\begin{aligned} \partial_0 (m; \gamma_1, \dots, \gamma_n) &= (m\gamma_1; \gamma_2 \dots \gamma_n) \\ \partial_i (m; \gamma_1, \dots, \gamma_n) &= (m; \gamma_1 \dots, \gamma_i \circ \gamma_{i+1}, \dots, \gamma_n);\ i > 0 \end{aligned}$$

and $*\Gamma$ by the corresponding simplicial set, where M is equal to a single point $*$:

(13) $$*\Gamma\colon * \leftleftarrows \Gamma \leftleftarrows \Gamma \times \Gamma \ldots,$$

Both of these are examples of simplicial manifolds in the sense that each is a collection of manifolds $\{X_k\}$ $k = 0, \ldots$ between which the usual semisimplicial structure maps are given by C^∞ maps. To any such "semi-simplicial" manifold

$$X\colon X_0 \leftleftarrows X_1 \leftleftarrows X_2$$

one associates a geometric realisation:

$$|X| = (\amalg X_k \times \Delta^k)/\sim,$$

obtained from the disjoint union of the $X_k \times \Delta^k$ where Δ^k is the k-simplex, by identifying $X_k \times \partial\Delta^k$ with the appropriate subsets of $X_{k-1} \times \Delta^{k-1}$.

Applied to (12) and (13) this construction then yields a map

(14) $$\begin{array}{c} |M\Gamma| \\ \downarrow \pi \\ |*\Gamma| \end{array}$$

which represents the associated M-bundle over the classifying space of Γ.

Now the point I wish to make is that the notions of Foliation, de Rham Theory, basic and Riemann connections all extend to this category, and provided one uses the Thom-Whitney-Sullivan-Dupont model for this extended de Rham theory so does the characteristic homomorphism $\varphi_*(F)$. To make this plausible let me say a few words concerning each one of these steps.

A foliation F on X, is simply a foliation F_k on each X_k which is preserved by the structure maps. For example the foliation $F\Gamma$ on $M \cdot \Gamma$ is represented by the trivial foliation

$$F_k\colon 0 \to T\{M \times \Gamma^{(k)}\} \to T\{M \times \Gamma^{(k)}\} \to 0.$$

of M by points, so that the normal bundle to F_k corresponds to $T\{M \times \Gamma^{(k)}\}$.

For the De Rahm theory of $|X|$ one uses the ring $\Omega^*(|X|)$ of *compatible forms on* $\amalg X_k \times \Delta^k$. Thus an r-form in $\Omega^*(|X|)$ is a function, which assigns to each k an r-form $\theta(k) \in \Omega^*(X_k \times \Delta^k)$ so that in the diagram

$$\begin{array}{ccc} \Omega^*(X_k \times \Delta^{k-1}) & \xleftarrow{1 \otimes i^*} & \Omega^*(X_k \times \Delta^k) \\ {\scriptstyle \partial_i^* \otimes 1} \downarrow & & \\ \Omega^*(X_{k-1} \times \Delta^{k-1}) & & \end{array}$$

one has the relation

(15) $$(1 \times i^*)\,\theta(k) = (\partial_i{}^* \times 1)\,\theta(k-1)\,.$$

Here of course i is any inclusion of a face in Δ_k and ∂_i the corresponding structure map of X_k to X_{k-1}.

This de Rahm theory is clearly anticommutative and computes the cohomology of $|X|$ under mild hypotheses on X:

$$H^*(|X|) = H\{\Omega^*(|X|)\}\,,$$

as was shown recently by Dupont [8]. Actually there is a more economical model for $H^*(|X|)$, namely the double complex

$$\Omega^{**}(X) = \oplus\, \Omega^p(X_q)$$

with differential $d \pm \delta$ where δ in the alternating sum of the structure maps, but this complex is not anticommutative and therefore not so suitable for our purposes. (See [8] or [4] for the De Rahm theorem in this context).

The final step is now to define basic and Riemann connections in this extended situation, and this is again all done by a compatability condition.

Thus if

$$F_k\colon 0 \to E_k \to TX_k \to Q_k \to 0$$

represents F on X_k then a basic connection for Q, is a compatible collection of basic connections for the pull-back of Q_k to $X_k \times \Delta^k$.

For instance in our situation, such a compatible collection can be constructed as follows:

Let ∇ be any connection on M. On the component $M \times f_1 \times \ldots \times f_k \times \Delta^k$ define $\nabla(k)$ by

(16) $$\nabla(k) = \nabla + x_i\,\xi_i\,,$$

where the ξ_i are defined as in (2) and the x_i are the usual baricentric coordinates of Δ^k. This is then a compatible collection as easily checked.

One proceeds similarly with the Riemann connections on Q although it is harder to give explicit formulas in that case and for this reason I know of no really attractive recepee for the classes involving the higher h_i's. For the $h_i\,S^\alpha$ one gets by because of the following stratagem:

Let θ be a volume on M. Then θ admits a natural compatible extension to $\Omega^1(|M \cdot \Gamma|)$ by setting

(17) $$\theta(f_1, \ldots, f_k) = u(\bar{f}_1)^{x1} \cdot u(\bar{f}_2)^{x2} \ldots u(\bar{f}_k) \cdot {}^{xk}\,\theta$$

where the x^i are again the baricentric coordinates on Δ^k, and the $\mu(f_i)$ are defined as in (3). (See [2] for some of the details and examples.)

So much then for the first question I raised. Concerning the second one, André Haefliger is reporting in detail so let me here just summarise the situation.

The proper formulation of the question is really in the frame work of Gelfand-Fuks. One seeks the smooth cohomology of the group of diffeomorphisms of M. Thus one sets $D = \text{Diff}(M)$ and tries to compute the cohomology of the complex of "smooth" Eilenberg cochains on D. Now the final answer for this problem, on which André Haefliger and I are working at the moment, seems to be the following one: There is a natural bundle $E(M)$ over M, on which D acts. This group therefore also acts on the space $S(E)$ of sections of E. Hence we may finally construct the homotopy quotient.

(18) $$S(E)/D = S(E) \times_D U(D)$$

that is, the bundle associated to the action of D on $S(E)$ over the classifying space of D. The cohomology of this space we believe gives the smooth cohomology of D.

(19) $$H^*_{\text{smooth}}(D) \simeq H^*\big(S(E)/D\big).$$

Of course this formula is of use only if D and its action on $S(E)$ are well understood. For instance for $M = S^1$, E is given by $S^3 \times S^1$, D by S^1 and the action of D on E by the translation. Hence

$$S(E) = \text{Map}(S^1, S^3)$$

and for $S(E)/D$ one finds—as André just tought me last week:

(20) $$H^*\big(S(E)/D\big) \simeq \mathbf{R}[\omega, e] \quad \omega \cdot e = 0$$

where $\omega, e \in H^2$.

Thus for the circle every smooth class of $\text{Diff}(S^1)$ is a power of ω—and this is just the $h_1 c_1$ of our discussion, or of one other class, e, which is the usual Euler class of the circle bundle $S^1 \cdot \Gamma$ over $*\Gamma$; $\Gamma = \text{Diff}(S^1)$ in the discrete topology. By a different argument Gelfand and Fuks had already determined this case a long time ago, but they made an error in the ring structure, and it is in any case interesting to verify the answer in this manner.

For the 2-sphere (18) gives an algorithm as I mentioned earlier and which I hope André is discussing in greater detail in his lecture. In any case one finds many classes besides those generated by the "foliation classes" and most likely the smooth cohomology of Diff (S^2), though finitely generated in each dimension is not finitely generated as a ring.

Finally a word concerning the last question I raised. The first example of a nontrivial foliation class, was given by Roussarie and Godbillon-Vey [7]. In our frame work, their example is as follows: Let $\Gamma \subset SL(2, \mathbf{R})$ be a discrete subgroup with compact fundamental domain. These exist—indeed every fundamental group of a Riemann surface of genus $\geqslant 2$, can be realised in this manner. Now let P be the subgroup of triangular matrixes in $G = SL(2, \mathbf{R})$. Then

$$G/P = S^1,$$

and hence Γ acts naturally on S^1. Now it turns out that both ω and $e \in H^2(\Gamma)$ are nontrivial for this action, on the other hand they are proportional, for any such "homogenous" example.

Since the paper of G.-V., this homogenous case has been studied by many people, notably Kamber and Tondeur [10], and quite recently (Unpublished) Baker. André Haefliger and I in our original note [3] were most probably the first to note some extensions of the example, however one of our assertions there, our independence theorem, was based on a misconception.

Let me try to rectify the matter by stating here the true fact, from which our independence assertion was incorrectly deduced. This theorem gives a sort of proportionality result for the classes $h_1 c_\alpha$.

Consider then the general case of a connected semi-simple Lie group G and let P be a parabolic subgroup of G, with dim $G/P = n$. Then after complexification $G^{\mathbf{C}}/P^{\mathbf{C}}$ is a *compact complex analytic* n-manifold and hence has well defined Chern classes and Chern numbers.

Our proportionality theorem then asserts that:

PROPOSITION. *For any* $\Gamma \subset G$, *the classes* $h_1 c^\alpha(F\Gamma) \in H^{2n+1}(|G/P \cdot \Gamma|)$ *are proportional to the Chern numbers* c^α *of* $G^{\mathbf{C}}/P^{\mathbf{C}}$. *Thus for any two multi-indexes* α *and* β

$$h_1 c^\alpha(F\,\Gamma) \cdot c^\beta = h_1 c^\beta(F\,\Gamma) \cdot c^\alpha . \tag{21}$$

Unfortunately this of course does not imply that the proportionality factor does not vanish, and that is alas, what happens most of the time in these examples.

Kamber and Tondeur, and as I said quite recently Baker, have explored this homogenous case in great detail and have found quite a few independent classes by this means, see [10] and [9]. On the other hand no "stable class" has ever been detected by a homogenous example.

Using the Lefschetz-Hyperplane Theorem for $G^{\mathbf{C}}/P^{\mathbf{C}}$, Haefliger and I have quite recently been able to explain this. Indeed one has the following quite general fact:

THEOREM. *All stable classes of* $F\Gamma$ *in* $H^*(|G/P\cdot\Gamma|)$ *vanish.*

Here P is parabolic in G, and $\Gamma \subset G$. Presumably the same is true in all homogenous cases but this is not clear to us so far.

The master of non homogenous actions has been first and foremost Thurston, and last year Heitch has been able to considerably extend Thurston's computations [9]. I will not be able to report on this work here, except to state what Thurston's constructions imply for the smooth cohomology of Diff (S^1). He shows first of all that e and ω are independent, and in fact he constructs a smooth family of actions of the group

$$\Pi = \{X, Y, U, V \mid [X, Y] = [U, V]\}$$

on S^1 for which the class ω varies continuously. See [2] for details in this context.) He also constructed examples to show that $\omega^n \neq 0$. On the other hand we still have no example for which e^n, $n \geqslant 2$ is nonzero.

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