

## 2. Determination of the extreme rays of W

Objektyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **23 (1977)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **22.09.2024**

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2. DETERMINATION OF THE EXTREME RAYS OF  $W$

Inspired by [6] we consider the Green's function

$$G(x, t) = \begin{cases} (1-x)t & \text{for } 0 \leq t < x \leq 1, \\ (1-t)x & \text{for } 0 \leq x \leq t \leq 1. \end{cases}$$

If  $\varphi$  is a continuous function on  $[0, 1]$  the unique solution  $f \in C([0, 1]) \cap C^2(]0, 1[)$  to the equations

$$(2.1) \quad f'' = -\varphi \text{ in } ]0, 1[, \quad f(0) = f(1) = 0$$

is

$$(2.2) \quad f(x) = \int_0^1 G(x, t) \varphi(t) dt.$$

The successive iterates of  $G$  are defined for  $x, t \in [0, 1]$  by the equations

$$G_1(x, t) = G(x, t)$$

$$G_n(x, t) = \int_0^1 G(x, y) G_{n-1}(y, t) dy, \quad n \geq 2.$$

It is clear that  $G_n(x, t) \geq 0$  for  $x, t \in [0, 1]$ .

We define recursively a sequence of polynomials  $(A_n)_{n \geq 0}^1$  by the requirement

$$(2.3) \quad A_0(x) = x, \quad A_n'' = -A_{n-1} \quad \text{and} \quad A_n(0) = A_n(1) = 0$$

for  $n \geq 1$ .

The polynomial  $A_n$  is of degree  $(2n + 1)$ , and we clearly have

$$(2.4) \quad A_n(x) = \int_0^1 G(x, t) A_{n-1}(t) dt = \int_0^1 G_n(x, t) t dt \quad \text{for}$$

$n \geq 1, x \in [0, 1].$

It follows that  $A_n \geq 0$  on  $[0, 1]$  for all  $n$ , and since  $(-1)^k A_n^{(2k)} = A_{n-k}$  for  $k \leq n$  we see that  $A_n \in W$ .

We recall that a ray  $\mathbf{R}_+x$  of a cone  $C$  is called *extreme*, if an equation  $x = f + g$  with  $f, g \in C$  is possible only if  $f, g \in \mathbf{R}_+x$ , cf. [3].

<sup>1</sup>) Our terminology is different from that of [6];  $(-1)^n A_n$  is equal to the  $n$ 'th Lidstone polynomial of [4] and [6].

PROPOSITION 2.1. *The polynomials  $A_n, n \geq 0$ , lie on extreme rays of  $W$ .*

*Proof.* If  $A_0 = f + g$  with  $f, g \in W$  we have  $0 = f'' + g''$ , but since  $f''$  and  $g''$  are both  $\leq 0$ , we conclude that  $f$  and  $g$  are affine. Furthermore, since  $f(0) = g(0) = 0$ , we conclude that  $f$  and  $g$  are proportional to  $A_0$ .

Suppose now that  $A_{n-1}, n \geq 1$ , lies on an extreme ray of  $W$ , and assume that  $A_n = f + g$  where  $f, g \in W$ . Then  $A_{n-1} = -f'' + (-g'')$ , and the induction hypothesis implies that  $-f''$  and  $-g''$  are proportional to  $A_{n-1}$ . Therefore we have  $f = \lambda A_n(x) + ax + b$  for certain numbers  $\lambda \geq 0, a, b$ . Since  $0 \leq f \leq A_n$ , we have  $f(0) = f(1) = 0$  which implies that  $a = b = 0$ . This proves that  $f$  (and similarly  $g$ ) are proportional to  $A_n$  which then lies on an extreme ray of  $W$ .

Since  $f \mapsto f^*$  is an affine isomorphism of  $W$  the polynomials  $A_n^*$  also lie on extreme rays of  $W$ . The following result is a special case of [6], Theorem 1.1.

PROPOSITION 2.2. *Every function  $f \in W$  can for  $n \geq 1$  be written as*

$$f(x) = \sum_{k=0}^{n-1} ((-1)^k f^{(2k)}(0) A_k^*(x) + (-1)^k f^{(2k)}(1) A_k(x)) + R_n(x),$$

where

$$R_n(x) = \int_0^1 G_n(x, t) (-1)^n f^{(2n)}(t) dt \in W.$$

*Proof.* For  $n = 1$  the formula is equivalent with

$$(2.5) \quad f(x) - f(0)(1-x) - f(1)x = R_1(x) = - \int_0^1 G(x, t) f''(t) dt,$$

which follows directly from (2.2), and it is clear that  $R_1 \in W$ .

Suppose now the formula holds for some  $n \geq 1$ . Applying (2.5) to  $(-1)^n f^{(2n)} \in W$  we get

$$\begin{aligned} (-1)^n f^{(2n)}(x) &= (-1)^n f^{(2n)}(0) A_0^*(x) + (-1)^n f^{(2n)}(1) A_0(x) \\ &+ \int_0^1 G(x, t) (-1)^{n+1} f^{(2n+2)}(t) dt, \end{aligned}$$

which substituted in the expression for  $R_n$  yields the formula for  $n+1$  because of (2.4).

To see that  $R_n \in W$  we notice that

$$(-1)^k R_n^{(2k)}(x) = \begin{cases} \int_0^1 G_{n-k}(x, t) (-1)^n f^{(2n)}(t) dt & \text{for } 0 \leq k \leq n-1 \\ (-1)^k f^{(2k)}(x) & \text{for } k \geq n. \end{cases}$$

The following lemma is easy to establish, but instead of giving the proof here we refer to [6].

LEMMA 2.3. *There exists a constant  $M > 0$  such that*

$$0 \leq \int_0^1 G_n(x, t) dt \leq \frac{M}{\pi^{2n}} \quad \text{for} \quad 0 \leq x \leq 1, \quad n \geq 1.$$

PROPOSITION 2.4. *The only functions  $f \in W$  satisfying  $f^{(2k)}(0) = f^{(2k)}(1) = 0$  for all  $k \geq 0$  are  $f(x) = a \sin(\pi x)$  with  $a \geq 0$ .*

*Proof.* Suppose  $f \in W$  satisfies  $f^{(2k)}(0) = f^{(2k)}(1) = 0$  for all  $k \geq 0$ . Defining  $a = \sup \{ \alpha \geq 0 \mid f - \alpha \sin(\pi x) \in W \}$ ,  $g = f - a \sin(\pi x)$  belongs to  $W$  because  $W$  is closed in  $\mathbf{R}^I$ . Furthermore

$$g^{(2k)}(0) = g^{(2k)}(1) = 0 \quad \text{for all } k \geq 0.$$

Let  $\varepsilon > 0$  be given. Since  $\varphi = g - \varepsilon \sin(\pi x) \notin W$ , there exist  $k \geq 0$  and  $x_0 \in ]0, 1[$  such that  $(-1)^k \varphi^{(2k)}(x_0) < 0$ , hence

$$(-1)^k g^{(2k)}(x_0) < \varepsilon \pi^{2k} \sin(\pi x_0).$$

By Lemma 1.2 (iii) applied to  $(-1)^k g^{(2k)}$  we get

$$(-1)^k g^{(2k)}(t) \leq \varepsilon \pi^{2k+1} \quad \text{for} \quad 0 < t < 1,$$

and therefore by Proposition 2.2 and Lemma 2.3 for  $0 < x < 1$

$$\begin{aligned} g(x) &= \int_0^1 G_k(x, t) (-1)^k g^{(2k)}(t) dt \leq \varepsilon \pi^{2k+1} \int_0^1 G_k(x, t) dt \\ &\leq \varepsilon M \pi. \end{aligned}$$

This proves that  $g$  is identically zero.

PROPOSITION 2.5. *The extreme rays of  $W$  are precisely the rays generated by  $\Lambda_n$  and  $\Lambda_n^*$ , where  $n \geq 0$ , and  $\sin(\pi x)$ .*

*Proof.* We first show that  $\sin(\pi x)$  lies on an extreme ray. If  $\sin(\pi x) = f + g$  where  $f, g \in W$ , we have  $f(0) = f(1) = g(0) = g(1) = 0$ . Differentiating  $2k$  times we similarly get  $f^{(2k)}(0) = f^{(2k)}(1) = g^{(2k)}(0) = g^{(2k)}(1) = 0$ , and it follows by Proposition 2.4 that  $f$  and  $g$  are proportional to  $\sin(\pi x)$ .

We finally have to show that an arbitrary extreme ray is generated by one of the above functions.

Assume that  $f \in W$  generates an extreme ray. If  $f^{(2k)}(0) = f^{(2k)}(1) = 0$  for all  $k \geq 0$  we already know by Proposition 2.4 that  $f$  is proportional to  $\sin(\pi x)$ . Otherwise let  $n$  be the smallest number  $\geq 0$  for which  $f^{(2n)}(0) \neq 0$  or  $f^{(2n)}(1) \neq 0$ . By Proposition 2.2 we then have

$$f(x) = (-1)^n f^{(2n)}(0) A_n^*(x) + (-1)^n f^{(2n)}(1) A_n(x) + R_{n+1}(x),$$

but since  $f$  lies on an extreme ray all three terms on the right-hand side lie on this ray.

If  $f^{(2n)}(0) \neq 0$  this shows that  $(-1)^n f^{(2n)}(1) A_n$  and  $R_{n+1}$  are proportional to  $A_n^*$ . Therefore  $f^{(2n)}(1) = 0$  and  $R_{n+1}^{(2n+2)} = f^{(2n+2)}$  is proportional to  $(A_n^*)^{(2n+2)} = 0$ , so that  $f^{(2n+2)} = 0$  and hence  $R_{n+1} = 0$  (cf. Proposition 2.2).

If  $f^{(2n)}(1) \neq 0$  we similarly get  $f^{(2n)}(0) = 0$  and  $R_{n+1} = 0$ . This shows that  $f$  lies on the ray generated by either  $A_n^*$  or  $A_n$ .

### 3. DETERMINATION OF A BASE FOR $W$

There are several ways of determining a base for  $W$ . We choose the following set

$$B = \left\{ f \in W \mid \int_0^1 f(x) \sin(\pi x) dx = 1 \right\}.$$

By Lemma 1.2 (ii) we get for  $f \in B$  and  $x_0 \in ]0, 1[$  that

$$1 \geq \frac{1}{\pi} f(x_0) \int_0^1 \sin^2(\pi x) dx = \frac{1}{2\pi} f(x_0),$$

so the functions in  $B$  are uniformly bounded by  $2\pi$ .

It is therefore clear that  $B$  is a compact convex base for  $W$ .

The extreme points of  $B$  are exactly the intersections between  $B$  and the extreme rays of  $W$ . We see that  $2 \sin(\pi x) \in B$ .

We claim that the following formulas hold, cf. [4]:

$$(3.1) \quad A_n^*(x) = \frac{2}{\pi^{2n+1}} \sum_{k=1}^{\infty} \frac{\sin(k\pi x)}{k^{2n+1}}, \quad n \geq 0, \quad x \in ]0, 1[ ,$$

$$(3.2) \quad A_n(x) = \frac{2}{\pi^{2n+1}} \sum_{k=1}^{\infty} (-1)^{k+1} \frac{\sin(k\pi x)}{k^{2n+1}}, \quad n \geq 0, \quad x \in ]0, 1[ .$$