EXTENSION AND LIFTING OF C^∞ WHITNEY FIELDS

EXTENSION AND LIFTING OF $\mathscr{C}^\infty$ WHITNEY FIELDS

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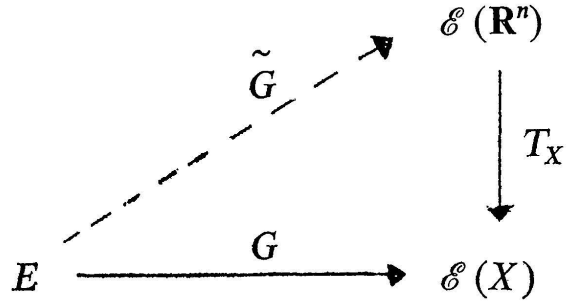
Whitney's Extension Theorem [10] provides a continuous linear extension operator from the space of $\mathscr{C}^m$ Whitney fields ($m < \infty$) on a closed subset X of $\mathbf{R}^n$, to the space of $\mathscr{C}^m$ functions on $\mathbf{R}^n$. For $\mathscr{C}^\infty$ Whitney fields, however, there does not in general exist a continuous linear extension operator [3]. Hence an *extension* problem arises: Under what conditions on X does there exist a continuous linear extension operator from the space $\mathscr{E}(X)$ of $\mathscr{C}^\infty$ Whitney fields on X to the space $\mathscr{E}(\mathbf{R}^n)$ of $\mathscr{C}^\infty$ functions? In fact we can formulate a more general *lifting* problem (cf. [4, Section 7]): Let T_X: $\mathscr{E}(\mathbf{R}^n) \to \mathscr{E}(X)$ be the canonical projection, associating to each $\mathscr{C}^\infty$ function its jet of infinite order on X. If E is a topological vector space, and G: $E \to \mathscr{E}(X)$ a continuous linear map, then under what conditions is there a continuous linear map $\tilde{G}: E \to \mathscr{E}(\mathbf{R}^n)$ such that the following diagram commutes?

(1)



By a lifting of G at the point $a \in X$, we will mean a continuous linear map $G_a: E \to \mathscr{E}(\mathbf{R}^n)$ such that $G(\xi) - T_X \circ G_a(\xi)$ is flat at a, for all $\xi \in E$. In this paper we prove that if E is a locally convex topological vector space, then a lifting $\tilde{G}$ of G exists provided that there exist pointwise lifts G_a: $E \to \mathscr{E}(\mathbf{R}^n)$, uniformly in $a \in X$. The uniformity of the pointwise lifts is the key ingredient in the proof, which is a simple argument using a Whitney partition of unity, analogous to the proof of Whitney's theorem in the $\mathscr{C}^m$ case ($m < \infty$). Nevertheless the result is a useful technical lemma.

Corollary 1 extends Mather's variant of Borel's Lemma [4, Section 7] to $\mathscr{C}^\infty$ Whitney fields on an arbitrary closed subset X of $\mathbf{R}^n$. Corollary 2, together with the well-known extension of $\mathscr{C}^\infty$ functions defined on a half-space [7], [6], provides a new proof of Stein's extension theorem for $\mathscr{C}^\infty$ functions on a domain with boundary which is Lipschitz of order 1 [8, Chapter VI, Theorem 5]. Corollary 2 is also used by one of the authors in [1], where Stein's theorem, for $\mathscr{C}^\infty$ Whitney fields, is extended to the case of a domain with boundary which is Lipschitz of any order, and this result is applied to the extension of $\mathscr{C}^\infty$ Whitney fields from a semianalytic subset $X \subset \mathbf{R}^n$ which is the closure of an open set.

Notation. Our notation is that of [9, Chapter IV]. If $k = (k_1, ..., k_n) \in \mathbf{N}^n$, $x = (x_1, ..., x_n) \in \mathbf{R}^n$, write $|k| = k_1 + ... + k_n$, $k! = k_1! ... k_n!$, $x^k = x_1^{k_1}, ..., x_n^{k_n}$. $\mathbf{N}^n$ is partially ordered by the relation: $k \leqslant l$ if and only if $k_j \leqslant l_j$, $j = 1, ..., n$. Write $\binom{l}{k} = \dfrac{l!}{k!(l-k)!}$ if $k \leqslant l$, $\binom{l}{k} = 0$ otherwise.

If Ω is an open subset of $\mathbf{R}^n$, then $\mathscr{E}(\Omega)$ denotes the space of $\mathscr{C}^\infty$ functions on Ω. $\mathscr{E}(\Omega)$ is a Fréchet space; its topology is defined by the seminorms

$$|f|_m^K = \sup_{\substack{x \in K \\ |k| \leq m}} \left| \frac{\partial^{|k|} f}{\partial x^k}(x) \right|,$$

where $m \in \mathbf{N}$ and $K \subset \Omega$ is compact.

Let X be a closed subset of Ω. A *jet of infinite order* on X is a sequence of continuous functions $F = (F^k)_{k \in \mathbf{N}^n}$ on X. $J(X)$ denotes the space of such jets. Write $|F|_m^K = \sup\limits_{\substack{x \in K \\ |k| \leq m}} |F^k(x)|$, and $F(x) = F^0(x)$, $x \in X$.

There is a linear map $J: \mathscr{E}(\Omega) \to J(X)$, associating to each $f \in \mathscr{E}(\Omega)$ the jet $J(f) = \left(\dfrac{\partial^{|k|} f}{\partial x^k} \Big| X \right)_{k \in \mathbf{N}^n}$. For each $k \in \mathbf{N}^n$, there is a linear map D^k: $J(X) \to J(X)$, defined by $D^k F = (F^{k+l})_{l \in \mathbf{N}^n}$. We also denote by D^k the map of $\mathscr{E}(\Omega)$ to itself, given by $D^k f = \dfrac{\partial^{|k|} f}{\partial x^k}$. This should cause no confusion since $D^k \circ J = J \circ D^k$.

If $a \in X$, $m \in \mathbf{N}$, $F \in J(X)$, then the *Taylor polynomial of order* m *of* F *at* a is the polynomial

$$T_a^m F(x) = \sum_{|k| \leq m} \frac{F^k(a)}{k!} (x-a)^k$$

of degree $\leqslant m$. Define $R_a^m F = F - J(T_a^m F)$, so that

$$(R_a^m F)^k(x) = F^k(x) - \sum_{|l| \leq m - |k|} \frac{F^{k+l}(a)}{l!}(x-a)^l$$

if $|k| \leqslant m$. Note that $D^k \circ R_a^m F(a) = (R_a^m F)^k(a) = 0, |k| \leqslant m$.

We say that $F \in J(X)$ is a *Whitney field of class* $\mathscr{C}^\infty$ on X if for each $m \in \mathbf{N}$, $|k| \leqslant m$:

$$(R_x^m F)^k(y) = o(|x-y|^{m-|k|})$$

as $|x - y| \to 0$, $x, y \in X$. $\mathscr{E}(X) \subset J(X)$ denotes the subspace of Whitney fields of class $\mathscr{C}^\infty$. $\mathscr{E}(X)$ is a Fréchet space, with the seminorms

$$\| F \|_m^K = | F |_m^K + \sup_{\substack{x, y \in K \\ x \neq y \\ |k| \leq m}} \frac{|(R_x^m F)^k(y)|}{|x-y|^{m-|k|}},$$

where $m \in \mathbf{N}$ and $K \subset X$ is compact.

Remarks 1. If $F \in J(\Omega)$, and for all $x \in \mathbf{R}^n$, $m \in \mathbf{N}$, $|k| \leqslant m$ we have

$$\lim_{y \to x} \frac{|(R_x^m F)^k(y)|}{|y-x|^{m-|k|}} = 0,$$

then there exists $f \in \mathscr{E}(\Omega)$ such that $F = J(f)$. This simple converse of Taylor's Theorem shows, in particular, that the two spaces we have denoted $\mathscr{E}(\Omega)$ are equivalent. On $\mathscr{E}(\Omega)$, the topologies defined by the seminorms $|\cdot|_m^K$, $\|\cdot\|_m^K$ are equivalent (by the Open Mapping Theorem).

2. The norms $|\cdot|_m^K$, $\|\cdot\|_m^K$ are not in general equivalent. They are, however, if the compact set K is connected by rectifiable arcs, and the geodesic distance on K is equivalent to the Euclidean distance (e.g. if K is convex) [9, Chapter IV, Proposition 2.6].

THEOREM. *Let X be a closed subset of $\mathbf{R}^n$, and E a topological vector space, topologized by a family of seminorms $\|\cdot\|_{\lambda \in \Lambda}$. Let $G: E \to \mathscr{E}(X)$ be a continuous linear map. Suppose that for each $a \in X$, there is a continuous linear map $G_a: E \to \mathscr{E}(\mathbf{R}^n)$ such that*

a) $G_a(\xi)^k(a) = G(\xi)^k(a)$ *for all* $\xi \in E$, $k \in \mathbf{N}^n$;

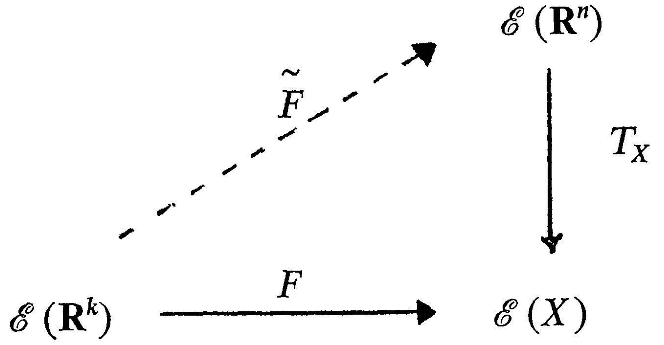
b) *for each $m \in \mathbf{N}$ and $L \subset \mathbf{R}^n$ compact, there exists $\lambda = \lambda(m, L) \in \Lambda$ and a constant $c = c(m, L)$ such that for all $\xi \in E$,*

(2) $$| G_a(\xi) |_m^L \leqslant c(m, L) \| \xi \|_{\lambda(m,L)}.$$

Then there exists a continuous linear map $\tilde{G}: E \to \mathscr{E}(\mathbf{R}^n)$ such that $\tilde{G}(\xi) | X = G(\xi)$, $\xi \in E$; i.e. the diagram (1) commutes.

To state Corollary 1, let X be a closed subset of $\mathbf{R}^n$, and $F\colon \mathscr{E}(\mathbf{R}^k) \to \mathscr{E}(X)$ a continuous linear map. As in [4, Section 7], we say F is *null* at $x \in \mathbf{R}^k$ if there exists a neighbourhood U of x such that if $f \in \mathscr{E}(\mathbf{R}^k)$ and $\operatorname{supp} f \subset U$, then $F(f) = 0$. The *support* of F is the complement of the set of points where F is null. Clearly $\operatorname{supp} F$ is closed.

COROLLARY 1. *If F has compact support, then there is a continuous linear map $\tilde{F}\colon \mathscr{E}(\mathbf{R}^k) \to \mathscr{E}(\mathbf{R}^n)$ such that $\tilde{F}(f) \mid X = F(f)$ for all $f \in E(\mathbf{R}^k)$; i.e. the following diagram commutes:*



Proof. It suffices to assume $X = K$, a compact subset of $\mathbf{R}^n$. Let $a \in K$. Mather's variant of Borel's Lemma [4, Section 7] provides a continuous linear map $F_a\colon \mathscr{E}(\mathbf{R}^k) \to \mathscr{E}(\mathbf{R}^n)$ such that $F(f) - T_X \circ F_a(f)$ is flat at a, for all $f \in \mathscr{E}(\mathbf{R}^k)$. Let L be a cube in $\mathbf{R}^k$ such that $\operatorname{supp} F \subset \operatorname{Int} L$. For each $r \in \mathbf{N}$, there exists $s(r) \in \mathbf{N}$ and a constant $c(r)$, such that for all $a \in K$,

$$\sup_{|k|=r} |F(f)^k(a)| \leqslant |F(f)|_r^K \leqslant c(r) \|f\|_{s(r)}^L .$$

The uniformity condition (2) for the pointwise lifts F_a then follows from Mather's estimates in [4]. Hence Corollary 1 follows from the Theorem, with the pointwise lifts given by the maps F_a.

Remark 3. If Y is a closed subspace of $\mathbf{R}^k$ for which there exists a continuous linear extension operator $\mathscr{E}(Y) \to \mathscr{E}(\mathbf{R}^k)$, then Corollary 1 holds more generally with $\mathscr{E}(\mathbf{R}^k)$ replaced by $\mathscr{E}(Y)$.

COROLLARY 2. *Let X be a closed subset of $\mathbf{R}^n$. Suppose that for each $a \in X$, there is a continuous linear map $W_a\colon \mathscr{E}(X) \to \mathscr{E}(\mathbf{R}^n)$ such that*

a) *$W_a(F)^k(a) = F^k(a)$ for all $F \in \mathscr{E}(X)$ and $k \in \mathbf{N}^n$;*

b) *for each $m \in \mathbf{N}^n$ and $L \subset \mathbf{R}^n$ compact, there exists $\lambda = \lambda(m, L) \in \mathbf{N}$, $K = K(m, L) \subset X$ compact, and a constant $c = c(m, L)$, such that for all $F \in \mathscr{E}(X)$,*

$$|W_a(F)|_m^L \leqslant c \|F\|_\lambda^K .$$

Then there exists a continuous linear map $W: \mathscr{E}(X) \to \mathscr{E}(\mathbf{R}^n)$ *such that* $W(F) \mid X = F$ *for all* $F \in \mathscr{E}(X)$.

This extension result follows immediately from the Theorem, with G given by the identity map of $\mathscr{E}(X)$.

Remarks 4. Corollary 2 may be used to prove Stein's extension theorem [8, Chapter VI, Theorem 5] for $\mathscr{C}^\infty$ functions. Let $y = \phi(x_1, ..., x_n)$ be a continuous function which satisfies the Lipschitz condition

$$(3) \qquad | \phi(x) - \phi(x') | \leqslant M \, | x - x' |$$

for all $x, x' \in \mathbf{R}^n$. We consider extension of $\mathscr{C}^\infty$ Whitney fields from the closed set

$$X = \{ (x, y) \in \mathbf{R}^{n+1} \mid y \geqslant \phi(x) \} .$$

Let Γ be the closed half-cone defined by $y \geqslant M(|x_1| + ... + |x_n|)$, and let $\Gamma(a) = a + \Gamma$ for any $a \in \mathbf{R}^{n+1}$. The Lipschitz condition (3) implies that $\Gamma(a) \subset X$ for any $a \in X$. Since Γ is defined by linear inequalities, Seeley's extension theorem [7] provides a continuous linear extension operator $S': \mathscr{E}(\Gamma) \to \mathscr{E}(\mathbf{R}^{n+1})$. Let $\rho: \mathbf{R}^{n+1} \to \mathbf{R}$ be a compactly supported $\mathscr{C}^\infty$ function which equals 1 in a neighborhood of 0. Define a continuous linear operator $S: \mathscr{E}(\Gamma) \to \mathscr{E}(\mathbf{R}^{n+1})$ by $S(F) = S'(\rho \cdot F)$, $F \in \mathscr{E}(\Gamma)$. The operators $W_a: \mathscr{E}(\Gamma(a)) \to \mathscr{E}(\mathbf{R}^{n+1})$, obtained by translating S to $\Gamma(a)$ for each $a \in X$, provide the pointwise extensions needed to apply Corollary 2.

5. Let $\mathscr{E}_p$ be the ring of germs at $0 \in \mathbf{R}^p$ of $\mathscr{C}^\infty$ functions, and $\mathfrak{m}$ its maximal ideal. Let $\phi: \mathbf{R}^n \to \mathbf{R}^p$ be a $\mathscr{C}^\infty$ map such that $\phi(0) = 0$. Then ϕ induces a ring homomorphism $\phi^*: \mathscr{E}(\mathbf{R}^p) \to \mathscr{E}(\mathbf{R}^n)$, defined by $\phi^*(f) = f \circ \phi$, $f \in \mathscr{E}(\mathbf{R}^p)$. We also denote by ϕ^* the induced homomorphism $\phi^*: \mathscr{E}_p \to \mathscr{E}_n$. We say ϕ is *finite* at 0 if $\mathscr{E}_n / \phi^*(\mathfrak{m}) \cdot \mathscr{E}_n$ is a finite dimensional real vector space. Let $b_1, ..., b_k \in \mathscr{E}(\mathbf{R}^n)$ represent a basis of this vector space; we take $b_1 \equiv 1$. By the Malgrange Preparation Theorem [9, Chapter IX, Theorem 3.2], the germs of $b_1, ..., b_k$ at 0 generate $\mathscr{E}_n$ over $\mathscr{E}_p$; i.e. for all $f \in \mathscr{E}(\mathbf{R}^n)$, there exist $g_1, ..., g_k \in \mathscr{E}(\mathbf{R}^p)$ such that $f = \sum_{j=1}^k \phi^*(g_j) \cdot b_j$ in some neighborhood of 0. A careful study of Mather's proof of this result ([5, Section 6] or [9, Chapter IX, Section 3]) shows, in fact, that there exist a neighborhood U of 0 in $\mathbf{R}^n$, and continuous linear operators $G_j: \mathscr{E}(\mathbf{R}^n) \to \mathscr{E}(\mathbf{R}^p)$, $j = 1, ..., k$, such that $f = \sum_{j=1}^k (\phi^* \circ G_j(f)) \cdot b_j$ in U, for all $f \in \mathscr{E}(\mathbf{R}^n)$.

Consider a $\mathscr{C}^\infty$ map $\phi: \mathbf{R}^n \to \mathbf{R}^n$ such that $\phi(0) = 0$. Let X, X' be closed subsets of $\mathbf{R}^n$ containing 0, such that $\phi(X') = X$. Suppose there is a continuous linear operator $W'\colon \mathcal{E}(X') \to \mathcal{E}(\mathbf{R}^n)$ such that $g - T_{X'} \circ W'(g)$ is flat at 0, for all $g \in \mathcal{E}(\mathbf{R}^n)$. If ϕ is finite at 0, then there exists a continuous linear operator $W\colon \mathcal{E}(X) \to \mathcal{E}(\mathbf{R}^n)$ such that $f - T_X \circ W(f)$ is flat at 0, for all $f \in \mathcal{E}(\mathbf{R}^n)$.

To see this, choose $b_j \in \mathcal{E}(\mathbf{R}^n)$ and $G_j\colon \mathcal{E}(\mathbf{R}^n) \to \mathcal{E}(\mathbf{R}^n)$, $j = 1, \ldots, k$, as above. Let $W = G_1 \circ W' \circ \phi^*$. That $f - T_X \circ W(f)$ is flat at 0, $f \in \mathcal{E}(\mathbf{R}^n)$, follows from the fact that for all $g \in \mathcal{E}(\mathbf{R}^n)$, the jets of $G_j(g)$ at 0, $j = 1, \ldots, k$, are uniquely determined by that of g (by [2, Proposition 5.2]). This remark might be useful in constructing the pointwise extensions needed to apply Corollary 2.

Proof of the Theorem. By an easy partition of unity argument, it suffices to assume $X = K$, a compact subset of $\mathbf{R}^n$. Let $\{ \Phi_i \mid i \in I\}$ be a Whitney partition of unity on $\mathbf{R}^n - K$ (as in [9, Chapter IV, Lemma 2.1]); i.e. a family of functions $\Phi_i \in \mathcal{E}(\mathbf{R}^n - K)$ satisfying the following conditions:

i) $\{ \operatorname{supp} \Phi_i \mid i \in I\}$ is a locally finite family. If $N(x)$ is the number of $\operatorname{supp} \Phi_i$ to which x belongs, then $N(x) \leqslant 4^n$.

ii) $\Phi_i \geqslant 0$ for all $i \in I$. $\Sigma_{i \in I} \Phi_i(x) = 1$ for all $x \in \mathbf{R}^n - K$.

iii) $2d(\operatorname{supp} \Phi_i, K) \geqslant \operatorname{diam}(\operatorname{supp} \Phi_i)$ for all $i \in I$.

iv) There exists a constant C_k, depending only on k and n, such that for all $x \in \mathbf{R}^n - K$,

$$| D^k \Phi_i(x) | \leqslant C_k \left(1 + \frac{1}{d(x, K)^{|k|}}\right).$$

Let $F = G(\xi) \in \mathcal{E}(K)$. For each $i \in I$, choose a point $a_i \in K$ such that $d(\operatorname{supp} \Phi_i, K) = d(\operatorname{supp} \Phi_i, a_i)$. Define $f = \tilde{G}(\xi) \in \mathcal{E}(\mathbf{R}^n)$ by

$$\begin{aligned} f(x) &= F^0(x), & x &\in K, \\ f(x) &= \sum_{i \in I} \Phi_i(x)\, G_{a_i}(\xi)(x), & x &\notin K. \end{aligned}$$

Then $f = \tilde{G}(\xi)$ clearly depends linearly on ξ, and is $\mathcal{C}^\infty$ on $\mathbf{R}^n - K$. We must show that f is $\mathcal{C}^\infty$, $D^k f \mid K = F^k$, and that $\tilde{G}$ is continuous. We write

$$\begin{aligned} f^k(x) &= F^k(x), & x &\in K, \\ f^k(x) &= D^k f(x), & x &\notin K. \end{aligned}$$

Let $m \in \mathbf{N}$, and L be a cube in $\mathbf{R}^n$ such that $K \subset \operatorname{Int} L$. There is a constant $c_1 = c_1(m, L)$ such that if $g \in \mathcal{E}(L)$, $| k | \leqslant m$, then

(4) $$|(R^m_a g)^k(x)| \leqslant c_1 \, | g |^L_m \cdot | x - a |^{m-|k|}$$

for all $a, x \in L$ (for example by [9, Chapter IV, (1.5.2)] and Remark 2 above).

Recall that a *modulus of continuity* is a continuous increasing function $\alpha\colon [0, \infty[\to [0, \infty[$ such that α is concave downwards and $\alpha(0) = 0$. By [9, Chapter IV, Remark 1.8] there exists a modulus of continuity α such that

(5) $$|(R^m_a F)^k(x)| \leqslant \alpha(|x-a|) \cdot |x - a|^{m-|k|}$$

if $a, x \in K$, $|k| \leqslant m$; and

(6) $$\begin{gathered} \alpha(t) = \alpha(\operatorname{diam} K) \quad \text{if} \quad t \geqslant \operatorname{diam} K, \\ \| F \|^K_m = | F |^K_m + \alpha(\operatorname{diam} K). \end{gathered}$$

It follows from (5) that if $a, b \in K$, $|k| \leqslant m$, then

(7) $$\begin{gathered} | D^k(T^m_a F)(x) - D^k(T^m_b F)(x) | \\ \leqslant 2^{m-|k|} e^{n/2} \alpha(|a-b|) \cdot (|x-a|^{m-|k|} + |x-b|^{m-|k|}) \end{gathered}$$

for all $x \in \mathbf{R}^n$ [9, Chapter IV, Remark 1.7].

Claim. There exists a constant $c_2 = c_2(m, L)$ such that if $|k| \leqslant m$, $a \in K$, $x \in L$, then

(8) $$\begin{gathered} |f^k(x) - D^k \circ G_a(\xi)(x)| \\ \leqslant c_2 \cdot (\|\xi\|_{\lambda(m,L)} + \alpha(|x-a|)) \cdot |x - a|^{m-|k|}. \end{gathered}$$

Once the claim is established, the proof of the theorem may be completed as follows. Let (j) be the multiindex whose j'*th* component is 1 and whose other components are 0. Let $k \in \mathbf{N}^n$, $a \in K$, $x \notin K$. Then

$$\begin{gathered} |f^k(x) - f^k(a) - \sum_{j=1}^{n} (x_j - a_j) \cdot f^{k+(j)}(a)| \\ \leqslant |f^k(x) - D^k \circ G_a(\xi)(x)| \\ + |D^k \circ G_a(\xi)(x) - D^k \circ G_a(\xi)(a) - \sum_{j=1}^{n} (x_j - a_j) \cdot D^{k+(j)} \circ G_a(\xi)(a)|. \end{gathered}$$

The second term in the right hand side is $o(|x-a|)$ since $G_a(\xi) \in \mathscr{E}(\mathbf{R}^n)$, while the first is $o(|x-a|)$ by the claim. Hence f^k is continuously differentiable, and $\dfrac{\partial f^k}{\partial x_j} = f^{k+(j)}$.

Let $\mu = \sup_{x \in L} d(x, K)$, $m \in \mathbf{N}$, $|k| \leqslant m$. Applying the claim to a point $x \in L$ and a point $a \in K$ such that $d(x, K) = d(x, a)$, we have

$$| D^k f(x) | \leqslant | D^k \circ G_a(\xi)(x) | + c_2 \cdot (\| \xi \|_{\lambda(m,L)} + \alpha(\mu)) \cdot \mu^{m-|k|}$$
$$\leqslant c \| \xi \|_{\lambda(m,L)} + c_2 \mu^{m-|k|} \cdot (\| \xi \|_{\lambda(m,L)} + \| G(\xi) \|_m^K)$$

by (8), (6). Hence there is a constant $c_3 = c_3(m, L)$ such that

$$| \tilde{G}(\xi) |_m^L \leqslant c_3 \cdot (\| \xi \|_{\lambda(m,L)} + \| G(\xi) \|_m^K) .$$

It follows that $\tilde{G}$ is continuous.

Proof of claim. We may assume $x \notin K$. Then

$$f(x) - G_a(\xi)(x) = \sum_{i \in I} \Phi_i(x) \cdot (G_{a_i}(\xi)(x) - G_a(\xi)(x)) .$$

Hence

$$f^k(x) - D^k \circ G_a(\xi)(x) = \sum_{l \leq k} \binom{k}{l} S_l(x) ,$$

where

$$S_l(x) = \sum_{i \in I} D^l \Phi_i(x) \cdot D^{k-l}(G_{a_i}(\xi)(x) - G_a(\xi)(x)) .$$

If $a, b \in K$, $|j| \leqslant m$, write

$$G_b(\xi)^j(x) - G_a(\xi)^j(x) = G_b(\xi)^j(x) - (T_b^m \circ G_b(\xi))^j(x)$$
$$+ (T_a^m \circ G_a(\xi))^j(x) - G_a(\xi)^j(x) + (T_b^m \circ G_b(\xi))^j(x) - (T_a^m \circ G_a(\xi))^j(x) .$$

Since $G_a(\xi)^j(a) = F^j(a)$, then

(9) $$| G_b(\xi)^j(x) - G_a(\xi)^j(x) |$$
$$\leqslant c_1 | G_b(\xi) |_m^L \cdot | x - b |^{m-j} + c_1 | G_a(\xi) |_m^L \cdot | x - a |^{m-|j|}$$
$$+ 2^{m-|j|} e^{n/2} \alpha(|a-b|) \cdot (|x-a|^{m-|j|} + |x-b|^{m-|j|})$$

by (4), (7)

$$\leqslant (c c_1 \| \xi \|_{\lambda(m,L)} + 2^{m-|j|} e^{n/2} \alpha(|a-b|)) \cdot (|x-a|^{m-|j|} + |x-b|^{m-|j|})$$

by (2).

To estimate $| S_0(x) |$, note that if $x \in \operatorname{supp} \Phi_i$, then $| x - a_i | \leqslant 3 | x-a |$ by iii), so that $| a - a_i | \leqslant 4 | x - a |$ and $\alpha(|a-a_i|) \leqslant 4\alpha(|x-a|)$. Hence

$$| S_0(x) | \leqslant 4^n (3^{m-|k|} + 1) \cdot (c c_1 \| \xi \|_{\lambda(m,L)} + 2^{m-|k|+2} e^{n/2} \alpha(|x-a|))$$
$$\cdot | x - a |^{m-|k|}$$

by i), ii).

Now consider $| S_l(x) |$, $l \neq 0$. For all $b \in K$,

$$S_l(x) = \sum_{i \in I} D^l \Phi_i(x) \cdot D^{k-l}(G_{a_i}(\xi)(x) - G_b(\xi)(x)) ,$$

since $\Sigma_{i \in I} D^l \Phi_i(x) = 0$. Choose b so that $|x - b| = d(x, K)$. As before, then $|x - a_i| \leqslant 3|x - b| \leqslant 3d(x, K)$, $|b - a_i| \leqslant 4d(x, K)$, $\alpha(|b - a_i|) \leqslant 4\alpha(d(x, K))$. By (9) and iv), there exist constants c', c'' depending only on m, L, such that

$$\begin{aligned} |S_l(x)| &\leqslant [c' \| \xi \|_{\lambda(m,L)} + c'' \alpha(d(x, K))] \cdot d(x, K)^{m - |k|} \\ &\leqslant (c' \| \xi \|_{\lambda(m,L)} + c'' \alpha(|x - a|)) \cdot |x - a|^{m - |k|}. \end{aligned}$$

This completes the proof of the claim, and the theorem.

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