# 4. Derivation of Siegel's Theorem

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$e_i = 1 / \# E(S_i)$, where $\#$ denotes cardinality. We now define the "number of representations of $T$ by the genus of $S$" as

$$A(\text{genus}(S), T) = \frac{e_1 A(S_1, T) + \ldots + e_h A(S_h, T)}{e_1 + \ldots + e_h}.$$

Now $S$ is a real symmetric matrix, and so we may view it as a point in $\mathbf{R}^{n_1}$, where $n_1 = n(n+1)/2$. Similarly, $T$ is a point in $\mathbf{R}^{m_1}$. Let $dt$ be the usual measure in $\mathbf{R}^{m_1}$, and let $dx$ be the usual measure in the real vector space of $m \times n$ matrices. Given $\varepsilon > 0$, let $B_\varepsilon$ denote the $\varepsilon$-neighborhood of $T$ in $\mathbf{R}^{m_1}$, and let $C_\varepsilon$ denote the set of $x \,\varepsilon\, M_{m \times n}(\mathbf{R})$ satisfying $S[x] \,\varepsilon\, B_\varepsilon$. Then $B_\varepsilon$ and $C_\varepsilon$ are open sets with compact closure, and the following limit is known to exist:

$$A_\infty(S, T) = \lim_{\varepsilon \to 0} \int_{C_\varepsilon} dx \Big/ \int_{B_\varepsilon} dt.$$

THEOREM (Siegel [4]). For $m - n \geqslant 3$,

$$(S) \qquad A(\text{genus}(S), T) = A_\infty(S, T) \lim_q \frac{A_q(S, T)}{q^{mn - (n+1)n/2}}.$$

## 4. Derivation of Siegel's Theorem

Let $G = \{g \in SL(m) : S[g] = S\}$, and let $X = \{x \in M_{m \times n} : S[x] = T\}$. If $m \geqslant 4$, both $G_{\mathbf{C}}$ and $G_{\xi\mathbf{C}}$ have fundamental groups of order 2. Condition (*c*) of § 2 is the classical Witt theorem for $(G, X)$. We assume that $X_{\mathbf{Q}}$ is nonempty.

We will show that (*A*) implies (*S*). This reduces Siegel's theorem to the computation of the Tamagawa number $\tau(G)$.

Let $\Phi_\infty$ = the constant function 1 on $X_{\mathbf{R}}$, and let $\Phi_p$ = the characteristic function of $X_{\mathbf{Z}_p}$ in $X_{\mathbf{Q}_p}$. Then $\Phi = \Phi_\infty \cdot \prod \Phi_p$ is the characteristic function of $X_{S_\infty} = X_{\mathbf{R}} \cdot \prod X_{\mathbf{Z}_p}$ in $X_A$. Because of the positive definiteness of $S$, $\Phi$ has compact support.

Consider the right-hand side of formula (*S*). Siegel has shown that there exists an algebraic gauge form $dx$ on $X$ such that $A_\infty(S, T) = \int_{X_{\mathbf{R}}} dx_\infty$, and

$$\lim_q \frac{A_q(S, T)}{q^{mn - (n+1)n/2}} = \prod_p \int_{X_{\mathbf{Z}_p}} dx_p,$$

where $dx$ and $dx_p$ are the positive measures induced on $X_{\mathbf{R}}$ and $X_{\mathbf{Q}_p}$ by $dx$.

It remains to identify the left-hand sides of $(A)$ and $(S)$. First we analyze the denominator of the left-hand side of $(A)$. Since $\tau(G) = \tau(G_\xi) = 2$ ([5]), this denominator is $\int_{G_A/G_{\mathbf{Q}}} dg$. Now $G_A$ admits a double coset decomposition

$$G_A = G_{S\infty}\sigma_1 G_{\mathbf{Q}} \dots G_{S\infty}\sigma_h G_{\mathbf{Q}}\,.$$

Then, following Tamagawa [5].

$$\begin{aligned}\int_{G_A/G_{\mathbf{Q}}} dg &= \sum_1^h \int_{(G_{S\infty}\sigma_i G_{\mathbf{Q}})/G_{\mathbf{Q}}} dg = \\ &= \sum_1^h \int_{(\sigma_i^{-1} G_{S\infty}\sigma_i G_{\mathbf{Q}})/G_{\mathbf{Q}}} dg \\ &= \sum_1^h \int_{(\sigma_i^{-1} G_{S\infty}\sigma_i)/G(\sigma_i)} dg\,,\end{aligned}$$

where $G(\sigma_i) = \sigma_i^{-1} G_{S\infty}\sigma_i \cap G_{\mathbf{Q}}$. This reduces to

$$\frac{\sum_1^h \int_{G_{S\infty}} dg}{\# G(\sigma_i)} = \int_{G_{S\infty}} dg \cdot \sum_1^h e_i\,.$$

A similar reduction applies to the numerator. First observe that for our choice of $\Phi$,

$$\begin{aligned}\sum_{x \varepsilon X_{\mathbf{Q}}} \Phi(gx) &= \#(X_{\mathbf{Q}} \cap g^{-1} X_{S\infty}) \\ &= \#(gX_{\mathbf{Q}} \cap X_{S\infty})\,.\end{aligned}$$

Then

$$\begin{aligned}\int_{G_A/G_{\mathbf{Q}}} \sum_{x\varepsilon X_{\mathbf{Q}}} \Phi(gx)\,dg &= \int_{G_A/G_{\mathbf{Q}}} \#(gX_{\mathbf{Q}} \cap X_{S\infty})\,dg \\ &= \sum_1^h \int_{(G_{S\infty}\sigma_i G_{\mathbf{Q}})/G_{\mathbf{Q}}} \#(gX_{\mathbf{Q}} \cap X_{S\infty})\,dg = \\ &= \sum_1^h \int_{(\sigma_i^{-1} G_{S\infty}\sigma_i G_{\mathbf{Q}})/G_{\mathbf{Q}}} \#(\sigma_i g X_{\mathbf{Q}} \cap X_{S\infty})\,dg \\ &= \sum_1^h \int_{(\sigma_i^{-1} G_{S\infty}\sigma_i)/G(\sigma_i)} \#(\sigma_i g X_{\mathbf{Q}} \cap X_{S\infty})\,dg \\ &= \sum_1^h e_i \int_{G_{S\infty}} \#(g\sigma_i X_{\mathbf{Q}} \cap X_{S\infty})\,dg \\ &= \sum_1^h e_i \int_{G_{S\infty}} \#(\sigma_i X_{\mathbf{Q}} \cap g^{-1} X_{S\infty})\,dg\end{aligned}$$

$$= \sum_1^h e_i \int_{GS\infty} \#(\sigma_i X_{\mathbf{Q}} \cap X_{S\infty})\, dg$$

$$= \int_{GS\infty} dg \cdot \sum_1^h e_i \#(\sigma_i X_{\mathbf{Q}} \cap X_{S\infty}) .$$

The left-hand side of $(A)$ therefore becomes

$$\frac{\int_{GS\infty} dg \cdot \sum_1^h e_i \#(\sigma_i X_{\mathbf{Q}} \cap X_{S\infty})}{\int_{GS\infty} dg \cdot \sum_1^h e_i} = \frac{\sum_1^h e_i \#(\sigma_i X_{\mathbf{Q}} \cap X_{S\infty})}{\sum_1^h e_i} .$$

The following result completes the identification of the left-hand side of $(A)$ with $A\big(\text{genus}(S), T\big)$.

PROPOSITION. $A(S_i, T) = \#(\sigma_i X_{\mathbf{Q}} \cap X_{S\infty})$.

Before giving the proof, we reinterpret the matrices $S_1, \ldots, S_h$. $G_A$ acts on the set of $\mathbf{Z}$-lattices in $\mathbf{Q}^n$ as follows: for $\sigma \varepsilon G_A$ and a lattice $L$, $\sigma * L$ is the unique lattice satisfying

$$(\sigma^* L) \otimes \mathbf{Z}_p = \sigma_p (L \otimes \mathbf{Z}_p) ,$$

for all $p$.

The matrix $S$ defines a quadratic form on $\mathbf{Q}^n$ by $q(x) = S[x]$. Consider the lattices $\sigma_1 * \mathbf{Z}^n, \ldots, \sigma_h * \mathbf{Z}^n$. In each lattice $\sigma_i * \mathbf{Z}^n$ choose a $\mathbf{Z}$-basis, and let $S_i$ be the matrix of $q$ with respect to this basis. Then $S_1, \ldots, S_h$ form a complete set of representatives of the $h$ classes in genus $(S)$ (see [7]).

Decomposing $(SL_m)_A = (SL_m)_{S\infty} (SL_m)_{\mathbf{Q}}$ we see that each $\sigma_i \varepsilon G_A$ can be written $\sigma_i = u_i a_i$, where $u_i \varepsilon (SL_m)_{S\infty}$, $a_i \varepsilon (SL_m)_{\mathbf{Q}}$. Then

$$\sigma_i^{-1} * \mathbf{Z}^m = a_i^{-1} u_i^{-1} * \mathbf{Z}^m = a_i^{-1} * (u_i^{-1} * \mathbf{Z}^m)$$
$$= a_i^{-1} * \mathbf{Z}^m = a_i^{-1} \mathbf{Z}^m .$$

Let $w_1, \ldots, w_m$ be the standard $\mathbf{Z}$-basis of $\mathbf{Z}^m$; then $a_i^{-1} w_1, \ldots, a_i^{-1} w_m$ is a $\mathbf{Z}$-basis of $\sigma_i^{-1} * \mathbf{Z}^m$. The matrix of $q$ with respect to this basis is

$$S_i = S[a_i^{-1}] = S[\sigma_i][a_i^{-1}] = S[\sigma_i a_i^{-1}] = S[u_i] .$$

LEMMA. Let $X_i = \{x \varepsilon M_{m \times n} \colon S_i[x] = T\}$. Then

(1) $(X_i)_{\mathbf{Q}} = a_i X_{\mathbf{Q}}$ ,

(2) $(X_i)_{S\infty} = u_i^{-1} X_{S\infty}$ .

*Proof of* (1): Let $x \in X_A$. $a_i x \in a_i X_{\mathbf{Q}} \Leftrightarrow x$ is **Q**-rational and $S[x] = T \Leftrightarrow a_i x$ is **Q**-rational and $S_i[a_i x] = T \Leftrightarrow a_i x \in (X_i)_{\mathbf{Q}}$.

The proof of (2) is similar.

Now we prove the proposition.

$$A(S_i, T) = \#(X_i)_{\mathbf{Z}} = \#((X_i)_{\mathbf{Q}} \cap (X_i)_{S\infty}) = \#(a_i X_{\mathbf{Q}} \cap u_i^{-1} X_{S\infty})$$
$$= \#(u_i a_i X_{\mathbf{Q}} \cap X_{S\infty}) = \#(\sigma_i X_{\mathbf{Q}} \cap X_{S\infty}).$$

## REFERENCES


[1] KNESER, M. Semi-simple algebraic groups. *Algebraic number theory.* Ed. J.W.S. Cassels and A. Frolich, Thompson Book Co. (1967).

[2] ONO, T. A mean value theorem in adele geometry. *Jour. Math. Soc. Japan, 20* (1968), pp. 275-288.

[3] SIEGEL, C.L. A mean value theorem in the geometry of numbers. *Ann. of Math. 46* (1945), pp. 340-347.

[4] —— Über die analytische Theorie der quadratischen Formen. *Ann. of Math. 26* (1935), pp. 527-606.

[5] TAMAGAWA, T. Adèles. *Proc. Symp. in Pure Math. IX, A.M.S.* (1966), pp. 113-121.

[6] WEIL, A. *Adeles and algebraic groups.* Lecture notes, Institute for Advanced Study, Princeton (1961).

[7] —— Sur la théorie des formes quadratiques. *Colloque sur la théorie des Groupes Algébriques*, Bruxelles (1962), pp. 9-22.

[8] —— Sur quelques résultats de Siegel. *Summa. Brasil. Math. 1* (1945-46), pp. 21-39.


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