

§3. The main lemma

Objektyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **23 (1977)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **22.09.2024**

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Let \mathcal{I}^n be the cube $\{0 \leq x_i \leq 1, i = 1, \dots, n\}$; $C(\mathcal{I}^n)$ -the space of all functions continuous on \mathcal{I}^n with the uniform norm; Φ -the space of functions continuous and non-decreasing on the segment \mathcal{I}^1 (with the uniform norm); $\Phi^k = \Phi \times \dots \times \Phi$ the k -th power of the space Φ .

THEOREM 3.2.1. *Let λ_p ($p=1, \dots, n$) be a collection of rationally independent constants. Then for quasi every collection $\{\varphi_1, \dots, \varphi_{2n+1}\} \in \Phi^{2n+1}$ it is true that any function $f \in C(\mathcal{I}^n)$ can be represented on \mathcal{I}^n in the form*

$$f(x) = \sum_{q=1}^{2n+1} g\left(\sum_{p=1}^n \lambda_p \varphi_q(x_p)\right),$$

where g is a continuous function.

§ 3. The main lemma

We fix a function $f \in C(\mathcal{I}^n)$, positive numbers λ_p ($p=1, \dots, n$) and a positive ε . We will denote by Ω_f the set of all collections $\{\varphi_1, \dots, \varphi_{2n+1}\} \in \Phi^{2n+1}$ for each of which there exists a continuous function h such that $\|h\| \leq \|f\|$ and $\|f(x) - \sum_{q=1}^{2n+1} h\left(\sum_{p=1}^n \lambda_p \varphi_q(x_p)\right)\| < (1-\varepsilon)\|f\|$. The latter inequality is strict and consequently the set Ω_f is open.

The idea of the construction is contained in the following statement.

LEMMA 3.3.1. *If $\|f\| \neq 0$, the numbers $\{\lambda_p\}$ are rationally independent, and $0 < \varepsilon < \frac{1}{2n+2}$, then the corresponding set Ω_f is everywhere dense in Φ^{2n+1} .*

Proof. Let us fix an open set $\Omega \subset \Phi^{2n+1}$ and prove that $\Omega \cap \Omega_f$ is not empty. This will imply that Ω_f is everywhere dense in Φ^{2n+1} .

We choose a number $\delta > 0$ and denote by $\mathcal{I}_q(j)$ the segment defined by the inequality

$$q \cdot \delta + (2n+1)j \cdot \delta \leq t \leq q \cdot \delta + (2n+1)j\delta + 2n\delta$$

($q=1, \dots, 2n+1, j$ is an integer).

The value δ will be determined below. Now we notice, firstly, that for any q the segments $\mathcal{I}_q(j)$ ($j=0, \pm 1, \pm 2$) are pairwise disjoint and every two consecutive segments are separated by an interval of length δ and, secondly, that, every point of the real axis belongs to at least $2n$ of the sets $\sum_j \mathcal{I}_q(j)$, ($q=1, \dots, 2n+1$).

We denote by $P_q(j_1, \dots, j_n)$ the cube

$$q\delta + (2n + 1)j_k\delta \leq x_k \leq q \cdot \delta + (2n + 1)j_k\delta + 2n\delta \quad (k = 1, \dots, n).$$

We emphasise that every point $x \in \mathcal{I}^n$ belongs to at least $n + 1$ of the sets

$\sum_{j_1, \dots, j_n} P_q(j_1, \dots, j_n)$ ($q = 1, \dots, 2n + 1$). We also remark that for any q the cubes $\{P_q(j_1, \dots, j_n)\}$ are pairwise disjoint.

We denote by Ω^* the subset of Φ^{2n+1} consisting of the collections $\varphi_1, \dots, \varphi_{2n+1}$ such that for every q the function φ_q is constant on every one of the segments $\{\mathcal{I}_q(j)\}$. We will assume that δ is so small that $\Omega^* \cap \Omega$ is not empty.

We choose a collection $\{\varphi_1, \dots, \varphi_{2n+1}\} \in \Omega^* \cap \Omega$. We will show that this collection belongs to Ω_f . We put $t_q \equiv \sum_{p=1}^n \lambda_p \varphi_q(x_p)$. Since the numbers $\{\lambda_p\}$ are rationally independent we can change the constants $\{\varphi_q(\mathcal{I}_q(j))\}$ slightly, so that the new values of $t_q(p_q(j_1, \dots, j_n))$ are pairwise different and the collection $\varphi_1, \dots, \varphi_{2n+1}$ remains in $\Omega^* \cap \Omega$.

We denote by $f_q(j_1, \dots, j_n)$ the value of the function f at the center of $P_q(j_1, \dots, j_n)$ and by h the function defined in the following way:

$$h(t_q(j_1, \dots, j_n)) = \frac{1}{2n + 1} f_q(j_1, \dots, j_n) \text{ outside the set } \cup_{q, j_1, \dots, j_n} t_q(j_1, \dots, j_n)$$

the function h is defined in such a way that it is continuous on the whole real axis and $\|h\| \leq \frac{1}{2n + 1} \|f\|$.

Now we estimate the function $|f - \sum_{q=1}^{2n+1} h(t_q)| = \left| \sum_{q=1}^{2n+1} \frac{f}{2n + 1} - h(t_q) \right|$. For any $x \in \mathcal{I}^n, q, j_1, \dots, j_n$

$$\begin{aligned} \left| \frac{f}{2n + 1} - h(t_q) \right| &\leq \frac{1}{2n + 1} \|f\| + \|h\| \leq \frac{1}{2n + 1} \|f\| + \frac{1}{2n + 1} \|f\| \\ &= \frac{2}{2n + 1} \|f\|. \end{aligned}$$

If $x \in P_q(j_1, \dots, j_n)$, then

$$\begin{aligned} &\left| \frac{f}{2n + 1} - h(t_q) \right| \\ &\leq \max_{q, j_1, \dots, j_n} \left| \max_{x \in p_q(j_1, \dots, j_n)} \frac{f(x)}{2n + 1} - \min_{x \in p_q(j_1, \dots, j_n)} \frac{f(x)}{2n + 1} \right| = \rho. \end{aligned}$$

We recall that every $x \in \mathcal{I}^n$ belongs to at least $n + 1$ of the cubes $\{P_q(j_1, \dots, j_n)\}$. Hence

$$\left| f - \sum_{q=1}^{2n+1} h(t_q) \right| \leq (n+1)\rho + n \frac{2}{2n+1} \|f\|.$$

But $\lim_{\delta \rightarrow 0} \rho = 0$, consequently for sufficiently small δ and $\varepsilon < \frac{1}{2n+2}$

$$\left| f - \sum_{q=1}^{2n+1} h(t_q) \right| < (1 - \varepsilon) \|f\|.$$

The lemma is proved.

§ 4. The proof of the theorem

We denote by F a countable set, everywhere dense in $C(\mathcal{J}^n)$. We choose ε satisfying the condition of lemma 3.3.1 and consider Ω_{f_k} ($f_k \in F$) corresponding to this ε and the collection λ_p mentioned in the theorem. The sets $\{\Omega_{f_k}\}$ are open and by lemma 3.3.1 they are everywhere dense in Φ^{2n+1} . Consequently, according to the definition, almost every element of Φ^{2n+1} belongs to $\Phi^* = \bigcap_{f_k \in F} \Omega_{f_k}$.

We fix a collection $\{\varphi_1, \dots, \varphi_{2n+1}\} \in \Phi^*$ and a function $f \in C(\mathcal{J}^n)$ and show that the desired representation of f takes place. If $f \equiv 0$ then as the function g we can take $g \equiv 0$. We will assume below that $f \not\equiv 0$. According to the definition of Ω_{f_k} there exists for any $f_k \in F$ a function h_k such that

$\left| f_k - \sum_{q=1}^{2n+1} h_k \left(\sum_{p=1}^n \lambda_p \varphi_q(x_p) \right) \right| \leq (1 - \varepsilon) \|f_k\|$. The set F is everywhere dense in $C(\mathcal{J}^n)$. Consequently for any $f \in C(\mathcal{J}^n)$ ($f \not\equiv 0$) there exists $h = \gamma(f)$ such that

$$\left| f - \sum_{q=1}^{2n+1} h \left(\sum_{p=1}^n \lambda_p \varphi_q(x_p) \right) \right| < \left(1 - \frac{\varepsilon}{2} \right) \|f\|.$$

We define the sequence of functions $\chi_0, \chi_1, \chi_2, \dots$ by the recurrent equalities

$$\chi_0 = f, \quad \chi_{k+1} = \chi_k - \sum_{q=1}^{2n+1} g_k \left(\sum_{p=1}^n \lambda_p \varphi_q(x_p) \right),$$

where $g_k = \gamma(\chi_k)$. The series $\sum_{k=0}^{\infty} g_k$ converges uniformly and consequently the function $g = \sum_{k=0}^{\infty} g_k$ is continuous and

$$f - \sum_{q=1}^{2n+1} g \left(\sum_{p=1}^n \lambda_p \varphi_q(x_p) \right) = 0.$$

The theorem is proved.