LEMMA 2.3.3. *If* $\frac{n}{s} > \frac{n'}{s'}$ *then for any natural* k *the set* $\Omega_k \cap C_s(\mathscr{I}^n)$ *is nowhere dense in* $C_s(\mathscr{I}^n)$.

By lemma 2.3.1 and the theorem 2.2.1 for any natural k $H_\varepsilon(\Omega_k) \leqslant C\left(\frac{1}{\varepsilon}\right)^{n'/s'}$, where C does not depend on ε. Hence, it follows from the inequality $\frac{n}{s} > \frac{n'}{s'}$ and lemma 2.3.2 that the set $\Omega_k \cap C_s(\mathscr{I}^n)$ is nowhere dense in $C_s(\mathscr{I}^n)$.

Now to prove the theorem we have to notice only that the set of functions from $C_s(\mathscr{I}^n)$ representable by superpositions coincides with $\bigcup_{k=1}^{\infty}(\Omega_k \cap C_s(\mathscr{I}^n))$. By lemma 2.3.3 the sets $\{\Omega_k \cap C_s(\mathscr{I}^n)\}$ are nowhere dense and consequently the set of not representable functions is a set of second category.

CHAPTER 3. — SUPERPOSITIONS OF CONTINUOUS FUNCTIONS

In this chapter we present the proof of the theorem of Kolmogorov given by Kahane [36]. This proof which is based on Baire's theory contains a minimum of concrete constructions and shows that there exists a wide choice of inner functions for Kolmogorov's formula.

§ 1. *Certain improvements of Kolmogorov's theorem*

By the theorem of Kolmogorov any function defined and continuous on the cube $\mathscr{I}^n$ can be represented as

$$f(x_1, \ldots, x_n) = \sum_{q=1}^{2n+1} g_q\Big(\sum_{p=1}^{n} \varphi_{p,q}(x_p)\Big),$$

where $\{\varphi_{p,q}\}$ are specially chosen continuous and monotonic functions which do not depend on f, and where $\{g_q\}$ are continuous functions.

Lorentz [12] has noticed that in the theorem of Kolmogorov the functions $\{g_q\}$ can be chosen independently of q. In fact, by adding constants to the functions $t_q = \sum_{p=1}^{n} \varphi_{p,q}(x_p)$ $(q = 1, \ldots, 2n+1)$ one can make the ranges of the functions pairwise disjoint and consequently the functions $\{ t_q \}$ can be considered as the restrictions of a single function $\{ g_q \}$.

Sprecher [40] has shown that the functions $\{ \varphi_{p,q} \}$ can be chosen in the form $\varphi_{p,q}(x_p) = \lambda_p \varphi_q(x_p)$ where $\{ \lambda_p \}$ are constants and $\{ \varphi_q \}$-are continuous monotonic functions.

Thus any continuous function can be represented as

$$f(x_1, \ldots, x_n) = \sum_{q=1}^{2n+1} g\left(\sum_{p=1}^{n} \lambda_p \varphi_q(x_p)\right),$$

where the constants $\{ \lambda_p \}$ and the continuous monotone functions $\{ \varphi_q \}$ do not depend on f, and where g is a continuous function.

Kahane [36] has shown that such a representation is possible with almost every collection of constants $\{ \lambda_p \}$ and "quasi every" collection of continuous functions $\{ \varphi_q \}$. The precise statement of this theorem will be given below. Here we consider some further results concerning the formula of Kolmogorov.

Doss [38] has shown that for any continuous monotonic functions $\varphi_{p,q}$ $(p=1, 2;\ q=1, 2, 3, 4)$ there exists a continuous function $f(x_1, x_2)$ of two variables not representable as a superposition of the form $\sum_{q=1}^{4} g_q \left(\sum_{p=1}^{2} \varphi_{p,q}(x_p)\right)$, where $\{ g_q \}$ are continuous functions.

Bassalygo [39] succeeded in showing that for any continuous functions $\varphi_i(x_1, x_2)$ $(i=1, 2, 3)$ there exists a continuous function $f(x_1, x_2)$ that is not equal to any superposition of the form $\sum_{i=1}^{3} g_i(\varphi_i(x_1, x_2))$, where $\{ g_i \}$ are continuous functions.

Tihomirov showed that Kolmogorov's theorem can be generalized as follows: for any compact K of dimension n there exists a homeomorphic embedding $\Psi(x) = \{ \Psi_1(x), \ldots, \Psi_{2n+1}(x) \}$, $x \in K$ into $(2n+1)$-dimensional euclidean space such that any continuous function $f(x)$ on K can be represented in the form $f(x) = \sum_{i=1}^{2n+1} g_i(\Psi_i(x))$, where $\{ g_i \}$ are continuous functions of one variable.

In the same paper [36] Kahane has shown that there exist complex numbers λ_p $(p=1, \ldots, n)$ and complex valued functions φ_q $(q=1, \ldots, 2n+1)$ possessing the following properties.

1. The function φ_q is a monotonic continuous transformation of the real axis onto the circle $|t| = 1$ $(q=1, \ldots, 2n+1)$.

2. The function $t_q = \sum_{p=1}^{n} \lambda_p \varphi_q (x_p)$ maps the cube $\mathcal{I}^n$ into the circle $|t| = 1$.

3. The transformation Ψ given by the equalities $t_q = \sum_{p=1}^{n} \lambda_p \varphi_q (x_p)$ $(q = 1, \ldots, 2n+1)$ is one-to-one on $\mathcal{I}^n$.

4. For any function f continuous on $\mathcal{I}^n$ there exists a function $g(z)$ continuous on the disk $|z| \leqslant 1$, holomorphic inside that disk, and such that $f = \sum_{q=1}^{2n+1} g\left(\sum_{p=1}^{n} \lambda_p \varphi_q (x_p)\right)$.

The transformation Ψ gives an embedding of the cube $\mathcal{I}^n$ into the torus $|t| = 1$ $(q = 1, \ldots, 2n+1)$ such that any function continuous on the cube $\tilde{\mathcal{I}}^n = \Psi(\mathcal{I}^n)$ is represented in the form $f(t_1, \ldots, t_{2n+1}) = \sum_{q=1}^{2n+1} g(t_q)$, where g is a function holomorphic in the unit disk. This means in particular that any function continuous on $\tilde{\mathcal{I}}^n$ has an analytic extension to the polydisk $|t_q| \leqslant 1$ $(q = 1, \ldots, 2n+1)$.

§ 2. *The theorem of Kahane*

Let M be a complete metric space. We recall that a set is called a set of second category if it is the intersection of a countable family of open sets which are everywhere dense in M. By the theorem of Baire in a complete metric space no set of second category is empty. The massivity of such sets is characterized by the fact that the intersection of a countable family of sets of second category is again a set of second category and consequently is not empty.

We will say that a statement is true for quasi every element of M if it is true for a set of elements of second category.

Let us consider an example. Let Φ be the space with uniform norm consisting of all functions continuous and non-decreasing on the segment $\mathcal{I}^1$ $(0 \leqslant t \leqslant 1)$. It can be shown easily that quasi every element of Φ is a strictly increasing function.

In fact, any strictly increasing function belongs to any set defined as $\varphi(r') < \varphi(r'')$, where $r' < r''$ are fixed rational numbers. Any set defined by an inequality of that type is open and everywhere dense in Φ, and the set of all such sets is countable.

Let $\mathscr{I}^n$ be the cube $\{ 0 \leqslant x_i \leqslant 1, i = 1, ..., n \}$; $C(\mathscr{I}^n)$-the space of all functions continuous on $\mathscr{I}^n$ with the uniform norm; Φ-the space of functions continuous and non-decreasing on the segment $\mathscr{I}^1$ (with the uniform norm); $\Phi^k = \Phi \times ... \times \Phi$ the k-th power of the space Φ.

THEOREM 3.2.1. *Let λ_p $(p=1, ..., n)$ be a collection of rationally independent constants. Then for quasi every collection $\{ \varphi_1, ..., \varphi_{2n+1} \} \in \Phi^{2n+1}$ it is true that any function $f \in C(\mathscr{I}^n)$ can be represented on $\mathscr{I}^n$ in the form*

$$f(x) = \sum_{q=1}^{2n+1} g\left(\sum_{p=1}^{n} \lambda_p \varphi_q(x_p) \right),$$

where g is a continuous function.

§ 3. *The main lemma*

We fix a function $f \in C(\mathscr{I}^n)$, positive numbers λ_p $(p=1, ..., n)$ and a positive ε. We will denote by Ω_f the set of all collections $\{ \varphi_1, ..., \varphi_{2n+1} \} \in \Phi^{2n+1}$ for each of which there exists a continuous function h such that $\| h \| \leqslant \| f \|$ and $\| f(x) - \sum_{q=1}^{2n+1} h\left(\sum_{p=1}^{n} \lambda_p \varphi_q(x_p) \right) \| < (1-\varepsilon) \| f \|$. The latter inequality is strict and consequently the set Ω_f is open.

The idea of the construction is contained in the following statement.

LEMMA 3.3.1. *If $\| f \| \neq 0$, the numbers $\{ \lambda_p \}$ are rationally independent, and $0 < \varepsilon < \dfrac{1}{2n+2}$, than the corresponding set Ω_f is everywhere dense in Φ^{2n+1}.*

Proof. Let us fix an open set $\Omega \subset \Phi^{2n+1}$ and prove that $\Omega \cap \Omega_f$ is not empty. This will imply that Ω_f is everywhere dense in Φ^{2n+1}.

We choose a number $\delta > 0$ and denote by $\mathscr{I}_q(j)$ the segment defined by the inequality

$$q \cdot \delta + (2n+1) j \cdot \delta \leqslant t \leqslant q \cdot \delta + (2n+1) j \delta + 2n\delta$$
$$(q=1, ..., 2n+1, j \text{ is an integer}).$$

The value δ will be determined below. Now we notice, firstly, that for any q the segments $\mathscr{I}_q(j)$ $(j=0, \pm 1, \pm 2)$ are pairwise desjoint and every two consecutive segments are separated by an interval of length δ and, secondly, that, every point of the real axis belongs to at least $2n$ of the sets $\sum_j \mathscr{I}_q(j)$, $(q=1, ..., 2n+1)$.

We denote by $P_q(j_1, \ldots, j_n)$ the cube

$$q\delta + (2n+1) j_k \delta \leqslant x_k \leqslant q \cdot \delta + (2n+1) j_k \delta + 2n\delta \quad (k = 1, \ldots, n).$$

We emphasise that every point $x \in \mathscr{I}^n$ belongs to at least $n + 1$ of the sets $\sum_{j_1, \ldots, j_n} P_q(j_1, \ldots, j_n)$ $(q = 1, \ldots, 2n+1)$. We also remark that for any q the cubes $\{ P_q(j_1, \ldots, j_n) \}$ are pairwise disjoint.

We denote by Ω^* the subset of Φ^{2n+1} consisting of the collections $\varphi_1, \ldots, \varphi_{2n+1}$ such that for every q the function φ_q is constant on every one of the segments $\{ \mathscr{I}_q(j) \}$. We will assume that δ is so small that $\Omega^* \cap \Omega$ is not empty.

We choose a collection $\{ \varphi_1, \ldots, \varphi_{2n+1} \} \in \Omega^* \cap \Omega$. We will show that this collection belongs to Ω_f. We put $t_q \equiv \sum_{p=1}^{n} \lambda_p \varphi_q(x_p)$. Since the numbers $\{ \lambda_p \}$ are rationally independent we can change the constants $\{ \varphi_q(\mathscr{I}_q(j)) \}$ slightly, so that the new values of $t_q(p_q(j_1, \ldots, j_n))$ are pairwise different and the collection $\varphi_1, \ldots, \varphi_{2n+1}$ remains in $\Omega^* \cap \Omega$.

We denote by $f_q(j_1, \ldots, j_n)$ the value of the function f at the center of $P_q(j_1, \ldots, j_n)$ and by h the function defined in the following way: $h(t_q(j_1, \ldots, j_n)) = \frac{1}{2n+1} f_q(j_1, \ldots, j_n)$ outside the set $\bigcup_{q, j_1, \ldots, j_n} t_q(j_1, \ldots, j_n)$ the function h is defined in such a way that it is continuous on the whole real axis and $\| h \| \leqslant \frac{1}{2n+1} \| f \|$.

Now we estimate the function $\left| f - \sum_{q=1}^{2n+1} h(t_q) \right| = \left| \sum_{q=1}^{2n+1} \frac{f}{2n+1} - h(t_q) \right|$. For any $x \in \mathscr{I}^n$, $q, j_1, \ldots, j_n$

$$\left| \frac{f}{2n+1} - h(t_q) \right| \leqslant \frac{1}{2n+1} \| f \| + \| h \| \leqslant \frac{1}{2n+1} \| f \| + \frac{1}{2n+1} \| f \|$$
$$= \frac{2}{2n+1} \| f \| .$$

If $x \in P_q(j_1, \ldots, j_n)$, then

$$\left| \frac{f}{2n+1} - h(t_q) \right|$$
$$\leqslant \max_{q, j_1, \ldots, j_n} \left| \max_{x \in p_q(j_1, \ldots, j_n)} \frac{f(x)}{2n+1} - \min_{x \in p_q(j_1, \ldots, j_n)} \frac{f(x)}{2n+1} \right| = \rho .$$

We recall that every $x \in \mathscr{I}^n$ belongs to at least $n + 1$ of the cubes $\{ P_q(j_1, \ldots, j_n) \}$. Hence

$$|f - \sum_{q=1}^{2n+1} h(t_q)| \leqslant (n+1)\rho + n\frac{2}{2n+1}\|f\| .$$

But $\lim_{\delta\to 0} \rho = 0$, consequently for sufficiently small δ and $\varepsilon < \dfrac{1}{2n+2}$

$$|f - \sum_{q=1}^{2n+1} h(t_q)| < (1-\varepsilon)\|f\| .$$

The lemma is proved.

§ 4. *The proof of the theorem*

We denote by F a countable set, everywhere dense in $C(\mathscr{I}^n)$. We choose ε satisfying the condition of lemma 3.3.1 and consider Ω_{f_k} $(f_k \in F)$ corresponding to this ε and the collection λ_p mentioned in the theorem. The sets $\{\Omega_{fk}\}$ are open and by lemma 3.3.1 they are everywhere dense in Φ^{2n+1}. Consequently, according to the definition, almost every element of Φ^{2n+1} belongs to $\Phi^* = \bigcap_{f_k \in F} \Omega_{f_k}$.

We fix a collection $\{\varphi_1, ..., \varphi_{2n+1}\} \in \Phi^*$ and a function $f \in C(\mathscr{I}^n)$ and show that the desired representation of f takes place. If $f \equiv 0$ then as the function g we can take $g \equiv 0$. We will assume below that $f \not\equiv 0$. According to the definition of Ω_{f_k} there exists for any $f_k \in F$ a function h_k such that $|f_k - \sum_{q=1}^{2n+1} h_k(\sum_{p=1}^{n} \lambda_p \varphi_q(x_p))| \leqslant (1-\varepsilon)\|f_k\|$. The set F is everywhere dense in $C(\mathscr{I}^n)$. Consequently for any $f \in C(\mathscr{I}^n)$ $(f \not\equiv 0)$ there exists $h = \gamma(f)$ such that

$$|f - \sum_{q=1}^{2n+1} h(\sum_{p=1}^{n} \lambda_p \varphi_q(x_p))| < \left(1 - \frac{\varepsilon}{2}\right)\|f\| .$$

We define the sequence of functions $\chi_0, \chi_1, \chi_2, ...$ by the recurrent equalities

$$\chi_0 = f, \quad \chi_{k+1} = \chi_k - \sum_{q=1}^{2n+1} g_k(\sum_{p=1}^{n} \lambda_p \varphi_q(x_p)),$$

where $g_k = \gamma(\chi_k)$. The series $\sum_{k=0}^{\infty} g_k$ converges uniformly and consequently the function $g = \sum_{k=0}^{\infty} g_k$ is continuous and

$$f - \sum_{q=1}^{2n+1} g(\sum_{p=1}^{n} \lambda_p \varphi_q(x_p)) = 0 .$$

The theorem is proved.