§5. Linear superpositions

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From the theorems of Kolmogorov and Bari it follows that each continuous function of n variables can be represented as a superposition of absolutely continuous monotonic functions of one variable and the operation of addition.

A detailed account of Kolmogorov's theorem is to be found in the surveys [9], [33]-[36]. The proof presented by Kahane is of special interest [36]. He does not attempt to construct the functions $\{ \varphi_{p,q} \}$ (as the proof of Kolmogorov does) but instead he shows by means of Baire's theorem, that most selections of increasing functions $\{ \varphi_{pq} \}$ will do. This approach also lead to other interesting results.

Fridman [37] showed that the inner functions $\{ \varphi_{pq} \}$ can be chosen from the class Lip 1. Kahane noticed that this follows directly from Kolmogorov's theorem. For any finite collection of continuous and monotone functions $\{ f_k(x) \}$ on the segment [0, 1] there exists a homeomorphism $x = \varphi(s)$ of the segment [0, 1] onto itself such that the functions $\{ g_k(s) = f_k(\varphi(s)) \}$ belong to the class Lip 1. The homeomorphism is taken as

$$s = \varphi^{-1}(x) = \varepsilon \big(x + \sum_k | f_k(x) - f_k(0) | \big) .$$

The constant ε is chosen to satisfy the condition $\varphi^{-1}(1) = 1$. By means of such homeomorphisms all inner functions in Kolmogorov's formula can be turned into functions satisfying the condition Lip 1.

There are some other improvements of Kolmogorov's theorem: Doss [38], Bassalygo [39], Lorentz [34], Sprecher [40] (see chapter 3, § 1). There are also many results concerning special types of superpositions (see [21], [33], [41]-[44]).

§ 5. *Linear superpositions*

We return again to superpositions of smooth functions.

One of the most interesting current problems on the subject of superpositions is the following: does there exist an analytic function of two variables that cannot be represented as a finite superposition of continuously differentiable (smooth) functions of one variable and the operation of addition?

Linear superpositions arise as a result of the following argument. Suppose that a function of two variables $f(x, y)$ is an s-fold superposition of certain smooth functions of one variable $\{ f_i(t) \}$ and the operation of addition. We vary this superposition, that is, we consider a superposition

$\tilde{f}(x, y)$ of the same form, but composed of the functions $\{f_i(t) + \varphi_i(t)\}$, where $\{\varphi_i(t)\}$ are small perturbations that are also smooth functions of one variable. Then the difference of these superpositions can be written in the form

$$\tilde{f}(x, y) - f(x, y) = \sum_{i=1}^{N} p_i(x, y)\,\varphi_i(q_i(x, y)) + o\,(\max_i \sup_t |\varphi_i(t)|), \quad \text{(III)}$$

where the functions $\{p_i(x, y)\}$ are expressed in terms of the original functions $\{f_i(t)\}$ and their derivatives, so that we can only say of them that they are continuous; $\{q_i(x, y)\}$ are expressed only in terms of the functions $\{f_i(t)\}$, hence they are continuously differentiable; the remainder term $o\,(\max_i \sup_t |\varphi_i(t)|)$ is an infinitely small quantity compared with $\max_i \sup_t |\varphi_i(t)|$, provided only that the functions $\left\{\frac{d\varphi_i}{dt}\right\}$ have some fixed modulus of continuity. Equation (III) gives some hope of reducing the general problem of superpositions of smooth functions to the determination of analytic functions not representable by superpositions of the form

$$\sum_{i=1}^{N} p_i(x, y)\,\varphi_i(q_i(x, y)), \quad \text{(IV)}$$

where $\{p_i(x, y)\}$ are preassigned continuous functions, $\{q_i(x, y)\}$ are preassigned continuously differentiable functions, and $\{\varphi_i(t)\}$ are arbitrary continuous functions of one variable.

Such superpositions are called linear, to emphasize the fact that the functions $\{p_i(x, y)\}$ and $\{q_i(x, y)\}$ are fixed and the superposition depends linearly on the variable functions $\{\varphi_i(t)\}$. We note that Kolmogorov's superpositions (I), (II) are also linear, since all $p_i \equiv 1$ and $q_i(x, y) = \alpha_i(x) + \beta_i(y)$ $(i = 1, 2, 3, 4, 5)$ are fixed continuous functions.

It is proved in [47], [48] that for any continuous functions $\{p_i(x, y)\}$ and continuously differentiable functions $\{q_i(x, y)\}$ there exists an analytic function of two variables not representable as a superposition of the form (IV). Henkin showed that the set of superpositions of the form (IV) is closed and consequently nowhere dense in the space of all continuous functions of two variables. Hence, in particular, it follows that there exists even a polynomial not representable as a superposition of the form (IV).

A comparison of these results with Kolmogorov's theorem leads to the conclusion that the inner functions of Kolmogorov's formula, although continuous, must inevitably be essentially non-smooth.

We note that the results mentioned above can be extended without any essential difficulties to superpositions of the form

$$\sum_{i=1}^{N} p_i(x_1, \ldots, x_n) f_i(q_i(x_1, \ldots, x_n)),$$

where $\{p_i\}$ are preassigned continuous functions, $\{q_i\}$ are preassigned smooth functions and $\{f_i\}$ are arbitrary continuous functions of one variable. But as it turns out this does not apply to superpositions of the form

$$\sum_{i=1}^{N} p_i(x_1, \ldots, x_n) f_i(q_{1,i}(x_1, \ldots, x_n), \ldots, q_{k,i}(x_1, \ldots, x_n)),$$

where $\{p_i\}$ are fixed continuous functions of n variables; and $\{q_{1i}\}, \ldots, \{q_{ki}\}$ are fixed smooth functions of n variables ($k<n$). Fridman answered that question only for $n = 3, 4$, $k = 2$ and $\{p_i\} \equiv 1$.

Also it is not known to what extent the problem of superpositions of smooth functions can be reduced to that of linear superpositions. "Such a reduction is proved only in the case of the so called stable" superpositions [10]. It turns out that not every analytic function of n variables can be represented by means of superpositions of smooth functions of a smaller number of variables it is assumed that the scheme is stable, i.e. for a small perturbation of a function represented the perturbations of the functions composing the superposition are comparatively small.

Chapter 2. — Superpositions of smooth functions

In this chapter we prove the existence of smooth functions of n variables ($n \geqslant 2$), not representable by superpositions of smooth functions of a smaller number of variables.

§ 1. *The notion of entropy*

We will denote by $C(\mathscr{I})$ the space of all functions defined on a set $\mathscr{I}$ and continuous on $\mathscr{I}$ (the norm is the maximum of the absolute value of the function). We fix a compact $F \subset C(\mathscr{I})$ and a positive number ε. A set $F^* \subset C(\mathscr{I})$ is called an ε-net of F if for any $f \in F$ there exists $f^* \subset F^*$