Chapter 1. — Survey of results

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existence of smooth functions of several variables not representable by superpositions of smooth functions of a smaller number of variables.

Bieberbach [5] attempted to prove that there exist continuous functions of three variables, not representable as a superposition of continuous functions of two variables. Not for nothing did Bieberbach call the 13-th Problem "unfortunate" (see [6]). Many years later, by the combined efforts of Kolmogorov [7], [9] and Arnol'd [8], the opposite was proved. So Hilbert's conjecture was shown to be false. By Kolmogorov's theorem any continuous function of several variables can be represented by means of a superposition of continuous functions of a single variable and the operation of addition.

Hilbert's 13-th problem gave rise to a great number of investigations in algebra and analysis, but the kernel of the problem never the less remains untouched. In this connection Lorentz [12] made an expressive analogy. The example of Peano of a mapping of an interval onto a square does not answer the question about the difference between an interval and a square. In the same way the theorem of Kolmogorov does not close the 13-th problem, but only makes it more interesting. It is known, for example, that superpositions of Kolmogorov's type, composed of smooth functions, do not even represent all analytic functions [48].

Thus, Hilbert's idea of proving the impossibility of solving the general equation of the 7-th degree by means of functions of only two variables can be developed in a more positive way. Results available at present do not contradict, for example, the possibility that the function $f(x, y, z)$ defined by the equation $f^7 + xf^3 + yf^2 + zf + 1 \equiv 0$ is not a finite superposition of analytic functions of two variables. On the other hand nobody has disproved that any algebraic function is a superposition of algebraic functions of a single variable and arithmetic operations.

This paper is a summary of the lectures given at the University of California in Los Angeles in April-May of 1977. Chapter I presents a survey of results, the remaining chapters are devoted to proofs.

Chapter 1. — Survey of results

The survey presented is based on the surveys [10]-[12], [33]-[35]. It also covers recent results:

Definition. We will say that a function $f = f(x_1, ..., x_n)$ is a superposition of the functions

$$\varphi^{(\alpha)}_{\beta_1, \beta_2, \ldots, \beta_\alpha}\left(U^{(\alpha+1)}_{\beta_1, \beta_2, \ldots, \beta_\alpha, 1}, U^{(\alpha+1)}_{\beta_1, \beta_2, \ldots, \beta_\alpha, 2}, \ldots, U^{(\alpha+1)}_{\beta_1, \beta_2, \ldots, \beta_\alpha, k}\right)$$
$$(\beta_i = 1, 2, \ldots, k, \;\; i = 1, \ldots, \alpha, \;\; \alpha = 0, 1, \ldots, s-1)$$

of k variables if f identically equals the function φ, defined by the equalities

$$\varphi = \varphi^{(0)}(U_1^{(1)}, U_2^{(1)}, \ldots, U_k^{(1)}),$$
$$U^{(\alpha)}_{\beta_1, \ldots, \beta_\alpha} = \varphi^{(\alpha)}_{\beta_1, \ldots, \beta_\alpha}\left(U^{(\alpha+1)}_{\beta_1, \beta_2, \ldots, \beta_\alpha, 1}, U^{(\alpha+1)}_{\beta_1, \beta_2, \ldots, \beta_\alpha, 2}, \ldots, U^{(\alpha+1)}_{\beta_1, \beta_2, \ldots, \beta_\alpha, k}\right)$$
$$\beta_i = 1, 2, \ldots, k, \;\; i = 1, 2, \ldots, \alpha, \;\; \alpha = 1, 2, \ldots, s-1,$$
$$U^{(s)}_{\beta_1, \beta_2, \ldots, \beta_s} = x_{j(\beta_1, \beta_2, \ldots, \beta_s)}.$$

The number s is called the order of superposition.

§ 1. *Superpositions of analytic functions*

In stating the 13-th Problem [1] Hilbert added that he had a rigorous proof of the fact that there exists an analytic function of three variables that cannot be obtained by a finite superposition of functions of only two arguments. Although he did not indicate exactly what kind of functions of two variables he had in mind, Hilbert was apparently thinking of analytic functions of two variables.

The existence of analytic functions of three variables not representable by means of superpositions of analytic functions of two variables is a simple fact and can be obtained from the following considerations. The partial derivatives of order k of a function represented by a superposition are defined by the derivatives of the functions composing the superposition. The number of different partial derivatives of order p of a function of two variables is equal to $\frac{p(p-1)}{1 \cdot 2}$. Consequently, the number of parameters defining the derivatives of order k of the superposition has order k^3 (s is fixed). On the other hand the number of different partial derivatives of order not greater than k for a function of three variables is of the order k^4. Hence for any s there exists a sufficiently large k such that one can find a polynomial of the k-th degree not representable by a superposition of order s of infinitely differentiable functions of two variables. The desired non-representable analytic function can be given as a sum of non-representable polynomials.

More general results in this direction were obtained by Ostrowski [2], who showed, in particular, that the analytic function of two arguments

$\xi(x, y) = \sum_{n=1}^{\infty} \frac{x^n}{n^y}$ is not a finite superposition of infinitely differentiable functions of one variable and algebraic functions of any number of variables.

The proof of this result is based on the fact that the function $\xi(x, y)$ does not satisfy any algebraic partial differential equation, that is, an equation of the form

$$\Phi\left(\xi, \frac{\partial \xi}{\partial x}, \frac{\partial \xi}{\partial y}, \ldots, \frac{\partial^{\mu+\lambda}\xi(x, y)}{\partial x^{\mu}\partial y^{\lambda}}\right) = 0, \quad \text{where} \quad \Phi$$

is a polynomial with constant coefficients in the function ξ and its partial derivatives up to a certain order. At the same, it is comparatively simple to prove that any function of two variables which is a finite superposition of infinitely differentiable functions of one variable and algebraic functions of any number of variables necessarily satisfies some algebraic partial differential equation. In the same paper, Ostrowski conjectured that the function $\xi(x, y)$ is not a superposition of continuous functions of one variable and algebraic functions of any number of variables (see the theorem of Kolmogorov [9]).

§ 2. *The problem of resolvents*

Algebraic equations up to the 4-th degree inclusive are soluble by radicals, that is, the roots of these equations can be represented as functions of the coefficients in the form of a superposition of arithmetic operations and functions of one variable of the form $\sqrt[n]{t}$ $(n=2, 3)$. The general equation of the 5-th degree, is insoluble by radicals, as Abel and Galois showed. But since the general equation of the 5-th degree may be reduced by algebraic substitutions to the form $x^5 + tx + 1 = 0$, containing a single parameter t, we may say that a root of the general equation of the 5-th degree is also represented as a function of the coefficients in the form of superpositions of arithmetic operations and algebraic functions of one variable. The problem of resolvents can be formulated in terms of superpositions in the following way: to find, for any number n, the smallest number k such that a root of the general equation of the n-th degree as a function of the coefficients is represented in the form of a superposition of algebraic functions of k variables. In [3] Hilbert conjectured that for $n = 6, 7, 8$ the number k is 2, 3, 4, respectively. Hilbert's result [3] for an equation of the 9-th degree was all the more unexpected: a root of the general equation of the 9-th degree is representable as a superposition of algebraic functions of four variables. Wiman [13], generalizing Hilbert's result, proved that $k \leqslant n - 5$ for any $n \geqslant 9$. As G. N. Chebotarev [14] observed, it can be proved by the same method that $k \leqslant n - 6$ for $n \geqslant 21$ and $k \leqslant n - 7$ for $n \geqslant 121$. A number of papers by N. G. Chebotarev [15] was devoted to the problem of resolvents. However, the basic Theorem turned out to be wrong (see [16]).

In correcting Chebotarev's paper Morosov found the right statements but his proofs also were not without essential gaps [17]. Nevertheless, in spite of the mistakes the papers of Chebotarev and Morosov have had a positive influence on subsequent authors.

Arnol'd [18] and Lin [17] have shown that the function $f_n = f(z_1, ..., z_n)$ which is the solution of the algebraic equation $f^n + z_1 f^{n-1} + z_2 f^{n-2} + ... + z_n = 0$ for $n \geqslant 3$ can not be strictly represented as a superposition of entire algebraic functions of a smaller number of variables and polynomials of any number of variables. Let us recall that a function $f = f(z_1, ..., z_k)$ is called an entire algebraic function if it satisfies an equation $f^m + p_1 f^{m-1} + ... + p_m = 0$, where $p_1, ..., p_m$ are polynomials in $z_1, ..., z_k$. The sentence "a function can not be strictly represented as a superposition" means in the case under consideration that every superposition representing the function must have unnecessary branches, i.e. the number of branches of any superposition must be at least $n + 1$. Using that theorem for $n = \{ 3, 4 \}$ we see that in spite of the fact that the equations of degree 3 and 4 are soluble by radicals they do not have strict representations. This explains in a sense why unnecessary roots appear when one uses Cardano's formulas.

Hovanski (see [19] and [20]) has shown that the solution of the equation $f^5 + xf^2 + yf + 1 = 0$ can not be represented by a superposition of entire algebraic functions of a single variable and polynomials in several variables. We recall that the Tschirnhaus transformation reduces the general equation of the 5-th degree to an equation with a single parameter, that is, the function of Hovanski is represented by a superposition of algebraic functions of a single variable and arithmetic operations. This counter example demonstrates that the restriction not to use the operation of division, is really strong.

We conclude the discussion of the problem of resolvents with a formulation of a well-known problem: is it possible to represent any algebraic function by means of a superposition of functions of a single variable and rational functions of any number of variables.

§ 3. *Superpositions of smooth functions and the theory of approximation*

In [4] it was proved that in the class of all S times continuously differentiable functions of n variables there exist some that cannot be represented as a finite superposition of functions for which the ratio of the number of arguments to the number of derivatives they have is strictly less than n/S.

This theorem shows that the ratio n/S can serve as a measure of the complexity of S times differentiable functions of n variables. The original proof of this theorem made use of the theory of multi-dimensional variations of sets and estimates of the number of ε-distant smooth functions (see [21], [22]). Kolmogorov [23] showed that the same result can be obtained using only estimates of the number of elements of ε-nets of functional compacts.

We denote by F_S^n the set of functions $f(x_1, x_2, ..., x_n)$ defined on an n-dimensional cube, whose partial derivatives up to order S inclusive are all continuous and bounded by some constant C. Let $N_\varepsilon(F_S^n)$ be the minimum number of spheres of radius ε in the space of all continuous functions by which the set F_S^n can be covered.

It turns out that

$$\lim_{\varepsilon \to 0} \frac{\log \log N_\varepsilon(F_s^n)}{\log\left(\frac{1}{\varepsilon}\right)} = \frac{n}{s}.$$

Hence it follows that if $n/S > n'/S'$, then the set of functions F_S^n is, in a certain sense, "more massive" than $F_{S'}^{n'}$.

If a consideration of the massivity of functional compacts does not give the answer then the problems remain open. For example, there is no answer to the question: is it possible to represent any analytic function of several variables by means of a superposition of smooth functions of a smaller number of variables.

The topic of superpositions led to a large number of papers in approximation theory. Here we formulate two results concerning non-linear approximations.

Let $\mathscr{I}^n$ be a cube $0 \leqslant x_i \leqslant 1$ $(i=1, ..., n)$; C—the space of all realvalued continuous functions defined on $\mathscr{I}^n$ with the uniform norm; F—a compact subset of C, Φ—a surface in C which consists of the functions represented in the form

$$\varphi = \frac{\sum_{\alpha_1+\alpha_2+\ldots+\alpha_p\leq k} a_1^{\alpha_1} . a_2^{\alpha_2} \ldots a_p^{\alpha_p} f_{\alpha_1,\ldots,\alpha_p}}{\sum_{\beta_1+\beta_2+\ldots+\beta_p\leq k} b_1^{\beta_1} . b_2^{\beta_2} \ldots b_p^{\beta_p} g_{\beta_1,\ldots,\beta_p}} .$$

where the natural numbers p and k and the collections $\{f_{\alpha_1,\ldots,\alpha_p} \in C\}$ and $\{g_{\beta_1,\ldots,\beta_p} \in C\}$ are fixed in advance and independent of φ, $\{a_i\}$ and $\{\beta_i\}$ are positive integers and the coefficients $\{a_i\}$ and $\{b_i\}$ defining the function φ can take arbitrary real values.

We remark that for $k = 1$ the class Φ can be turned into any of the usual classes in approximation theory by means of an appropriate choise of the number p and collections $\{f_{\alpha_1,\ldots,\alpha_p}\}$ and $\{g_{\beta_1,\ldots,\beta_p}\}$. For example it can be turned into the classes of polynomials or rational functions of a fixed degree.

We put $e_{pk}(F) = \sup_{f\in F} \inf_{\varphi\in\Phi} \| f - \varphi \|$. Estimates of e_{pk} for some functional compacts can be found in [21], [22], [24], [25]. Here are two examples of such estimates

$$1. \quad e_{pk}(F_s^n) \geqslant a\left(\frac{1}{p \log (k+1)}\right)^{s/n},$$

where $a > 0$ does not depend on p and k.

2. For the set F_{dc} consisting of all functions which have an analytic extension to some domain d in n-dimensional complex space bounded in modulus by some constant C the following inequality is valid

$$e_{pk}(F_{dc}) \geqslant bq^{\sqrt[n]{p\log(k+1)}},$$

where $b > 0$ and $0 < q < 1$ are constants independent of p and k.

Now there are more elementary proofs of these inequalities for $k = 1$ with precise estimates of the constant (see Erohin [26], Lorentz [24], Tihomirov [27], Shapiro [25]).

Let us clarify the meaning of these inequalities. We agree to characterize the complexity of any algorithm for the approximate calculation of functions firstly by the number of parameters used in the algorithm, and secondly by the complexity of the scheme of the calculation, for example, by the number of arithmetic operations required for the approximate calculation of functions by means of the given algorithm.

In the above-mentioned method of approximation of functions by functions from Φ the parameters are the numbers $\{a_i\}$ and $\{b_i\}$, and the number of arithmetic operations increases very rapidly as k increases. At the same time, from the inequalities mentioned above it follows that an increase in k leads to an insignificant improvement in the accuracy of the approximation. Hence, in a certain sense, a more economical approximation of functions is by means of expressions of the given form with $k = 1$, that is, by fractions of the form

$$\frac{\sum_{i=0}^{p} a_i f_i(x)}{\sum_{j=0}^{p} b_j g_j(x)}$$

The same inequalities with $k = 1$ show that there are no methods of approximating functions by fractions of the given form essentially better than the standard methods of approximating functions by algebraic (or trigonometric) polynomials.

§ 4. *Superpositions of continuous functions*

Kolmogorov's theorem on the possibility of representing continuous functions of n variables as superpositions of continuous functions of three variables was highly unexpected (see [7]).

In this paper Kolmogorov proves that on the n-dimensional cube $\mathscr{I}^n$ we can construct continuous functions $\varphi_i(x)$ $(i = 1, 2, \dots, n+1)$ such that any continuous function $f(x)$, defined on the cube $\mathscr{I}^n$, can be represented in the form

$$f(x) = \sum_{i=1}^{n+1} f_i(d_i(x)),$$

where $d_i(x)$ is a continuous mapping of $\mathscr{I}^n$ onto the one-dimensional tree [1]) D if the components of the level sets of the functions $\varphi_i(x)$, and $f_i(d_i)$ is a continuous function on the tree D_i. Since the trees $\{D_i\}$ can be embedded homeomorphically in the plane (see [30]), the functions $\{f_i(d_i(x))\}$ can be thought of as superpositions

$$\{f_i(u_i(x_1, x_2, \dots, x_n),\ v_i(x_1, x_2, \dots, x_n))\}$$

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[1]) Kronrod [29] has shown that the components of all possible level sets of any continuous function defined on $\mathscr{I}^n$ in a certain natural topology, form a tree, that is, a one-dimensional locally connected continuum, not containing homeomorphic images of circles. Kronrod calls this the "one-dimensional tree of the function".

where $\{f_i(u_i, v_i)\}$ are continuous functions of two variables, and $\{u_i(x)\}$ and $\{v_i(x)\}$ are fixed continuous functions of n variables. Kolmogorov derived from this the result that for $n \geqslant 4$ any continuous function of n variables can be represented by the following superposition of continuous functions of not more than $n - 1$ variables:

$$\sum_{i=1}^{n} f_i\big(u_i(x_1, x_2, \ldots, x_{n-1}),\ v_i(x_1, x_2, \ldots, x_{n-1}),\ x_n\big).$$

Arnol'd [8], [22] showed that, firstly, in Kolmogorov's construction [7] we can manage with functions $\{\varphi_i(x)\}$ whose one-dimensional trees $\{D_i\}$ have index at each branch point equal to 3, and, secondly, for any compact set F of functions defined on such a tree D, the given tree can be so placed in three-dimensional u, v, w-space that any continuous function $f(d) = f(u, v, w) \in F$ can be represented as the sum of functions of the coordinates, $f(u, v, w) = \varphi(u) + \psi(v) + \kappa(w)$. Hence it follows that any continuous function $f(x, y, z)$ of three variables can be represented as a superposition of the form $f(x, y, z) = \sum_{i=1}^{9} f_i(\varphi_i(x, y), z)$, where all the functions are continuous, and the functions $\{\varphi_i(x, y)\}$ can be regarded as fixed, when $f(x, y, z)$ is taken from a compact set. Thus, Arnol'd had the last word in refuting Hilbert's conjecture. At the same time Kolmogorov [9] obtained, in a certain sense, the definitive result in this direction.

Each continuous function of n variables, given on the unit cube in n-dimensional space, is representable as a superposition of the form

$$f(x_1, x_2, \ldots, x_n) = \sum_{q=1}^{2n+1} g_q\Big(\sum_{p=1}^{n} \varphi_{p,q}(x_p)\Big), \tag{I}$$

where all the functions are continuous, and moreover the functions $\{\varphi_{p,q}(x_p)\}$ are standard and monotonic.

In particular, each continuous function of two variables is representable in the form

$$f(x, y) = \sum_{i=1}^{5} f_i(\alpha_i(x) + \beta_i(y)). \tag{II}$$

Kolmogorov's theorem can be supplemented by the following result of Bari, which was obtained in connection with problems of Fourier series: any continuous function of one variable $f(t)$ can be represented in the form $f(t) = f_1(\varphi_1(t)) + f_2(\varphi_2(t)) + f_3(\varphi_3(t))$, where all the functions $\{f_i\}$ and $\{\varphi_i\}$ are absolutely continuous [32].

From the theorems of Kolmogorov and Bari it follows that each continuous function of n variables can be represented as a superposition of absolutely continuous monotonic functions of one variable and the operation of addition.

A detailed account of Kolmogorov's theorem is to be found in the surveys [9], [33]-[36]. The proof presented by Kahane is of special interest [36]. He does not attempt to construct the functions $\{ \varphi_{p,q} \}$ (as the proof of Kolmogorov does) but instead he shows by means of Baire's theorem, that most selections of increasing functions $\{ \varphi_{pq} \}$ will do. This approach also lead to other interesting results.

Fridman [37] showed that the inner functions $\{ \varphi_{pq} \}$ can be chosen from the class Lip 1. Kahane noticed that this follows directly from Kolmogorov's theorem. For any finite collection of continuous and monotone functions $\{ f_k (x) \}$ on the segment [0, 1] there exists a homeomorphism $x = \varphi (s)$ of the segment [0, 1] onto itself such that the functions $\{ g_k (s) = f_k (\varphi (s)) \}$ belong to the class Lip 1. The homeomorphism is taken as

$$s = \varphi^{-1} (x) = \varepsilon \bigl(x + \sum_k | f_k (x) - f_k (0) |\bigr) .$$

The constant ε is chosen to satisfy the condition $\varphi^{-1} (1) = 1$. By means of such homeomorphisms all inner functions in Kolmogorov's formula can be turned into functions satisfying the condition Lip 1.

There are some other improvements of Kolmogorov's theorem: Doss [38], Bassalygo [39], Lorentz [34], Sprecher [40] (see chapter 3, § 1). There are also many results concerning special types of superpositions (see [21], [33], [41]-[44]).

§ 5. *Linear superpositions*

We return again to superpositions of smooth functions.

One of the most interesting current problems on the subject of superpositions is the following: does there exist an analytic function of two variables that cannot be represented as a finite superposition of continuously differentiable (smooth) functions of one variable and the operation of addition?

Linear superpositions arise as a result of the following argument. Suppose that a function of two variables $f (x, y)$ is an s-fold superposition of certain smooth functions of one variable $\{ f_i (t) \}$ and the operation of addition. We vary this superposition, that is, we consider a superposition

$\tilde{f}(x, y)$ of the same form, but composed of the functions $\{f_i(t) + \varphi_i(t)\}$, where $\{\varphi_i(t)\}$ are small perturbations that are also smooth functions of one variable. Then the difference of these superpositions can be written in the form

$$\tilde{f}(x, y) - f(x, y) = \sum_{i=1}^{N} p_i(x, y)\,\varphi_i(q_i(x, y)) + o\,(\max_i \sup_t |\varphi_i(t)|), \quad \text{(III)}$$

where the functions $\{p_i(x, y)\}$ are expressed in terms of the original functions $\{f_i(t)\}$ and their derivatives, so that we can only say of them that they are continuous; $\{q_i(x, y)\}$ are expressed only in terms of the functions $\{f_i(t)\}$, hence they are continuously differentiable; the remainder term $o\,(\max_i \sup_t |\varphi_i(t)|)$ is an infinitely small quantity compared with $\max_i \sup_t |\varphi_i(t)|$, provided only that the functions $\left\{\frac{d\varphi_i}{dt}\right\}$ have some fixed modulus of continuity. Equation (III) gives some hope of reducing the general problem of superpositions of smooth functions to the determination of analytic functions not representable by superpositions of the form

$$\sum_{i=1}^{N} p_i(x, y)\,\varphi_i(q_i(x, y)), \quad \text{(IV)}$$

where $\{p_i(x, y)\}$ are preassigned continuous functions, $\{q_i(x, y)\}$ are preassigned continuously differentiable functions, and $\{\varphi_i(t)\}$ are arbitrary continuous functions of one variable.

Such superpositions are called linear, to emphasize the fact that the functions $\{p_i(x, y)\}$ and $\{q_i(x, y)\}$ are fixed and the superposition depends linearly on the variable functions $\{\varphi_i(t)\}$. We note that Kolmogorov's superpositions (I), (II) are also linear, since all $p_i \equiv 1$ and $q_i(x, y) = \alpha_i(x) + \beta_i(y)$ $(i = 1, 2, 3, 4, 5)$ are fixed continuous functions.

It is proved in [47], [48] that for any continuous functions $\{p_i(x, y)\}$ and continuously differentiable functions $\{q_i(x, y)\}$ there exists an analytic function of two variables not representable as a superposition of the form (IV). Henkin showed that the set of superpositions of the form (IV) is closed and consequently nowhere dense in the space of all continuous functions of two variables. Hence, in particular, it follows that there exists even a polynomial not representable as a superposition of the form (IV).

A comparison of these results with Kolmogorov's theorem leads to the conclusion that the inner functions of Kolmogorov's formula, although continuous, must inevitably be essentially non-smooth.

We note that the results mentioned above can be extended without any essential difficulties to superpositions of the form

$$\sum_{i=1}^{N} p_i(x_1, \ldots, x_n) f_i(q_i(x_1, \ldots, x_n)),$$

where $\{p_i\}$ are preassigned continuous functions, $\{q_i\}$ are preassigned smooth functions and $\{f_i\}$ are arbitrary continuous functions of one variable. But as it turns out this does not apply to superpositions of the form

$$\sum_{i=1}^{N} p_i(x_1, \ldots, x_n) f_i(q_{1,i}(x_1, \ldots, x_n), \ldots, q_{k,i}(x_1, \ldots, x_n)),$$

where $\{p_i\}$ are fixed continuous functions of n variables; and $\{q_{1i}\}, \ldots, \{q_{ki}\}$ are fixed smooth functions of n variables ($k < n$). Fridman answered that question only for $n = 3, 4$, $k = 2$ and $\{p_i\} \equiv 1$.

Also it is not known to what extent the problem of superpositions of smooth functions can be reduced to that of linear superpositions. "Such a reduction is proved only in the case of the so called stable" superpositions [10]. It turns out that not every analytic function of n variables can be represented by means of superpositions of smooth functions of a smaller number of variables it is assumed that the scheme is stable, i.e. for a small perturbation of a function represented the perturbations of the functions composing the superposition are comparatively small.

Chapter 2. — Superpositions of smooth functions

In this chapter we prove the existence of smooth functions of n variables ($n \geqslant 2$), not representable by superpositions of smooth functions of a smaller number of variables.

§ 1. *The notion of entropy*

We will denote by $C(\mathscr{I})$ the space of all functions defined on a set $\mathscr{I}$ and continuous on $\mathscr{I}$ (the norm is the maximum of the absolute value of the function). We fix a compact $F \subset C(\mathscr{I})$ and a positive number ε. A set $F^* \subset C(\mathscr{I})$ is called an ε-net of F if for any $f \in F$ there exists $f^* \subset F^*$