§5. Examples

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§5. Examples

Given an elliptic curve E in the form of a minimal model (1.1) or (1.2), one computes the bad primes by finding the prime divisors of the discriminant Δ. We can then apply the methods of the preceding sections to determine f_p and hence the type of reduction.

Example 5.1. Let E be given by $Y^2 = X^3 + X + 1$. This equation is minimal. The discriminant is $\Delta = -16(31)$, so E has bad reduction at $p = 2$ and $p = 31$. For $p = 2$, $C_{p-1} = C_1 = a_1 = 0$ so we have additive reduction at $p = 2$. For $p = 31$, we can apply Theorem 4.3 and Corollary 4.4. $f_p = \left(\frac{-2AB}{p}\right) = \left(\frac{-2}{31}\right) = -1$, so that E has non-split multiplicative reduction at $p = 31$. Alternatively, one may use Deuring's formula to compute C_{p-1}. A third possibility, of course, is to factor $X^3 + X + 1$ over $\mathbf{Z}/31\mathbf{Z}$ and then analyse (4.14). $c_4 = -48$.

Example 5.2. Let E be given by $Y^2 = X^3 + X - 1$. The equation is minimal and $\Delta = -16(31)$. We have additive reduction at $p = 2$ since $C_{p-1} = C_1 = a_1 = 0$. For $p = 31$, $f_p = \left(\frac{-2AB}{p}\right) = \left(\frac{2}{31}\right) = 1$, so that E has split multiplicative reduction at $p = 31$. $c_4 = -48$.

Remark. Comparing examples 5.1 and 5.2, one sees that c_4 is the same in both cases. However, 5.1. exhibits non-split multiplicative reduction at $p = 31$, while 5.2 exhibits split multiplicative reduction at the same prime.

Example 5.3. Let E be given by $Y^2 = X^3 + 7X + 5$. The equation is minimal and $\Delta = -16(23)(89)$. E has bad reduction at $p = 2, 23$, and 89. For $p = 2$, $C_{p-1} = C_1 = a_1 = 0$, so we have additive reduction at $p = 2$. For $p = 23$, we have $f_p = \left(\frac{-2AB}{p}\right) = \left(\frac{-70}{23}\right) = \left(\frac{-1}{23}\right) = -1$, so that E has non-split multiplicative reduction at $p = 23$. For $p = 89$, we have $f_p = \left(\frac{-2AB}{p}\right) = \left(\frac{19}{89}\right) = -1$, so that E has non-split multiplicative reduction at $p = 89$ as well.

Remark. The computation of the Legendre symbol is much easier to carry out in practice than either the computation of C_{p-1} via Deuring's formula or by searching for roots of the polynomial $X^3 + AX + B$.

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