§1. Introduction

A SIMPLE FORMULA
CONCERNING MULTIPLICATIVE REDUCTION
OF ELLIPTIC CURVES

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The purpose of this note is to show how the coefficients of the canonical invariant differential on an elliptic curve E defined over the field $\mathbf{Q}$ of rational numbers may be used to determine the type of reduction at a prime p where E has bad reduction. Simple and explicit formulas for the coefficients at these primes are obtained. This yields an easy method for calculating the local L-functions at the bad primes. To do this we use a theorem of Honda [2, 3] which says that the formal group F of the curve E is strongly isomorphic over $\mathbf{Z}$ to the formal group G associated to the global L-series of E. We then proceed to analyse the singularity of the reduced curve and obtain the desired formulas. In particular, let E be given by an affine equation $Y^2 = X^3 + AX + B$ with $A, B \in \mathbf{Z}$, which is minimal at a prime $p \geqslant 5$ where E has bad reduction. If $L_p(s) = (1 - f_p p^{-s})^{-1}$ is the local L-function at p, then $f_p = \left(\dfrac{-2AB}{p} \right)$ where $\left(\dfrac{}{p} \right)$ denotes the Legendre symbol at p. Formulas are given for $p = 2$ and $p = 3$ as well.

§1. Introduction

Let K be a field and let E be an elliptic curve defined over K, i.e. a non-singular projective curve of genus one defined over K together with a K-rational point e on E which acts as the identity element for the group law on E. Any such elliptic curve is isomorphic over K to an elliptic curve in the projective plane $\mathbf{P}^2$ defined by the equation

$$(1.1) \quad \overline{Y}^2 \overline{Z} + a_1 \overline{X}\,\overline{Y}\overline{Z} + a_3 \overline{Y}\overline{Z}^2 = \overline{X}^3 + a_2 \overline{X}^2 \overline{Z} + a_4 \overline{X}\overline{Z}^2 + a_6 \overline{Z}^3$$

with $a_i \in K$. The corresponding affine equation is

(1.2) $$Y^2 + a_1 XY + a_3 Y = X^3 + a_2 X^2 + a_4 X + a_6$$

The K-rational point $e = (0, 1, 0)$ is the identity element on this curve. From now on we shall assume that E is given by an equation of this form. Such an equation is called a *Weierstrass equation* for E. Given such an equation, it is usual to define the following invariants:

$$b_2 = a_1^2 + 4a_2, \qquad b_4 = a_1 a_3 + 2a_4, \qquad b_6 = a_3^2 + 4a_6,$$
$$b_8 = b_2 a_6 - a_1 a_3 a_4 + a_2 a_3^2 - a_4^2, \qquad c_4 = b_2^2 - 24 b_4,$$
$$c_6 = -b_2^3 + 36 b_2 b_4 - 216 b_6,$$

and

$$\Delta = -b_2^2 b_8 - 8 b_4^3 - 27 b_6^2 + 9 b_2 b_4 b_6.$$

We note that b_2 and b_4 correspond to Neron's $\bar{\alpha}$ and $\bar{\beta}$ [4, p. 450]. Δ is the *discriminant*. In general if we are given a curve defined by an equation of the form (1.1) or (1.2), it will have a singular point if and only if $\Delta = 0$.

Suppose now that v is a discrete valuation on K normalized so that $v(K^*) = \mathbf{Z}$. Let $R = \{\chi \in K \mid v(\chi) \geqslant 0\}$ be its valuation ring, $M = \{\chi \in K \mid v(\chi) > 0\}$ its maximal ideal, and $k = R/M$ the residue class field. Under the circumstances we may always assume that the coefficients a_i are actually in R. Among all Weierstrass equations with $a_i \in R$, one for which $v(\Delta)$ is minimal is called a *minimal Weierstrass equation* for E. By taking the residue classes of the a_i in k, we obtain from a minimal Weierstrass equation a curve defined over k, the *reduction* of E at v. If the reduction does not have singularities (i.e. $v(\Delta) = 0$), then the reduced curve is an elliptic curve and E is said to have *good reduction*. If the reduction has a singularity (it can have at most one such), then E is said to have *bad reduction*. The singularity may be either a cusp or a node. If the singularity is a cusp, E is said to have *additive reduction*. If the singularity is a node and the two tangents are rational over k, then E has *split multiplicative reduction*. If the singularity is a node and the two tangents are not rational over k, then E has *non-split multiplicative reduction*.

Suppose now that $K = \mathbf{Q}$. Each prime p induces a discrete valuation. Since $\mathbf{Z}$ is a principal ideal domain, it is possible to find a Weierstrass equation which is simultaneously minimal at all primes p, a *global minimal Weierstrass equation*. The primes where E has bad reduction are precisely those which divide Δ. For each prime p we define an integer f_p as follows: $f_p = 0$ if E has additive reduction at p, $f_p = 1$ if E has split multiplicative reduction at p, and $f_p = -1$ if E has non-split multiplicative reduction at p. If E has good reduction at p, let N_p denote the number of $\mathbf{Z}/p\mathbf{Z}$-rational

points of the reduced curve and let $f_p = 1 + p - N_p$. f_p is the trace of Frobenius at p. The *local L-function* $L_p(s)$ *of* E *at* p is defined as $L_p(s) = (1 - f_p p^{-s} + p^{1-2s})^{-1}$ if E has good reduction at p, and $L_p(s) = (1 - f_p p^{-s})^{-1}$ if E has bad reduction at p. The *global L-function* of E is $L(s) = \prod_p L_p(s)$.

If R is a commutative ring, then a (one-dimensional) *formal group* over R is a formal power series $F(X, Y) \in R[[X, Y]]$ in two variables such that $F(X, 0) = X$, $F(0, Y) = Y$, and $F(F(X, Y), Z) = F(X, F(Y, Z))$. Given the global L-function $L(s)$ of an elliptic curve defined over $\mathbf{Q}$, write $L(s) = \sum_{m=1}^{\infty} a_m m^{-s}$. If we set $g(X) = \sum_{m=1}^{\infty} (a_m/m) X^m$ and let $G(X, Y) = g^{-1}(g(X) + g(Y))$, then it can be shown that G is a formal group over $\mathbf{Z}$, the *formal group associated* to $L(s)$. On the other hand, there is another formal group with coefficients in $\mathbf{Z}$ which can be attached to an elliptic curve E defined over $\mathbf{Q}$. If we let $Z = -X/Y$, then Z is a uniformizing parameter in the local ring of E at e. The group law on E is given by a morphism $E \times E \to E$ in which $(e, e) \to e$. We thus have induced a natural homomorphism from the local ring of E at e to $E \times E$ at (e, e). If we complete each of these rings with respect to the topology induced by their respective maximal ideals, we obtain power series rings $\mathbf{Q}[[Z]], \mathbf{Q}[[Z_1, Z_2]]$ in one and two variables over $\mathbf{Q}$. The morphism $E \times E \to E$ induces a local $\mathbf{Q}$-algebra homomorphism $\mathbf{Q}[[Z]] \to \mathbf{Q}[[Z_1, Z_2]]$. The image $F(Z_1, Z_2)$ of Z is then a formal group due to the properties of the group law on E. F has its coefficients in $\mathbf{Z}$. Given two formal groups F and G defined over a commutative ring R, a *homomorphism* $f : F \to G$ *over* R consists of a formal power series $f(T) \in R[[T]]$ such that $f(0) = 0$ and $f(F(X, Y)) = G(f(X), f(Y))$. f is a *strong isomorphism* over R if in addition $f(X) \equiv X \pmod{\text{degree } 2}$ and there exists a homomorphism $g : G \to F$ over R such that $f \circ g = T$ and $g \circ f = T$. A fundamental result which we wish to use here is the following:

THEOREM 1.1 (Honda [2, 3]). Let E be an elliptic curve defined over $\mathbf{Q}$. The formal group F of E is strongly isomorphic over $\mathbf{Z}$ to the formal group G associated to the global L-function of E.

Among all the differentials on E, those which are translation-invariant form a one-dimensional vector space over $\mathbf{Q}$. Given such a differential ω we may expand ω/dZ as a power series in Z. The *canonical invariant differential* on E is the unique one which has 1 as the constant term in this

expansion. Let ω be the canonical invariant differential on E and write $\omega/dZ = \sum_{i=0}^{\infty} C_i Z^i$. It is an immediate consequence of Honda's theorem that $f_p \equiv C_{p-1} \pmod{p}$.

COROLLARY 1.2. Let E be an elliptic curve, and assume that E has bad reduction at a prime p. Then

(1) $C_{p-1} \equiv 0, 1, -1 \pmod{p}$.

(2) E has additive reduction at $p < = > C_{p-1} \equiv 0 \pmod{p}$.

(3) For $p > 2$, E has split multiplicative reduction at $p < = > C_{p-1} \equiv 1 \pmod{p}$.

(4) For $p > 2$, E has non-split multiplicative reduction at $p < = > C_{p-1} \equiv -1 \pmod{p}$.

Proof: Since $C_{p-1} \equiv f_p \pmod{p}$ and $f_p = 0, 1,$ or -1, the congruence class of C_{p-1} modulo p determines the reduction type uniquely as indicated except for $p = 2$.

From now on we shall assume that all curves and points are defined over **Q**, and that all Weierstrass equations are minimal. We wish to derive some simple arithmetical criteria for determining which of the three types of reduction occurs at a given prime p where E has bad reduction. From now on we shall assume that E has bad reduction at the prime p under discussion.

§2. THE CASE $p = 2$

For a curve given in the form (1.2), we have

(2.1) $$\omega = dX/(2Y + a_1 X + a_3)$$

Expressing X and Y in terms of Z and computing (cf. Tate [5] for the details), one obtains

(2.2) $$C_1 = a_1$$

THEOREM 2.1. Assume E has bad reduction at 2.

(1) E has additive reduction at $2 < = > a_1 \equiv 0 \pmod{2} < = > c_4 \equiv 0 \pmod{2}$.

(2) E has split multiplicative reduction at $2 < = > a_1 \equiv 1 \pmod{2}$ and $a_2 + a_3 \equiv 0 \pmod{2}$.