§0. Introduction

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ON THE VALUES AT NEGATIVE INTEGERS OF THE ZETA-FUNCTION OF A REAL QUADRATIC FIELD

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§0. INTRODUCTION

In this paper we will be interested in the numbers $\zeta_K(b)$, where K is a real quadratic field and b a negative odd integer. It has been known for some time [3] that these numbers are rational; indeed, this is true for K any totally real number field [5], [9]. They are interesting on the one hand because they generalize Bernoulli numbers (the special case $K = \mathbf{Q}$) and on the other because they reflect properties of the arithmetic of K. For example, there is a conjecture of Bass, Birch and Tate relating $\zeta_K(-1)$ to the "deviation from the Hasse principle" of K (= order of $\operatorname{Ker}(K_2 K \to \prod K_2 K_{\mathfrak{P}})$, with $K_{\mathfrak{P}}$ running over the completions of K). The value of $\zeta_K(b)$, and in particular the problem of estimating its denominator, is related to formulas for the "Euler characteristic" of certain arithmetic groups (see for instance [6]).

Our main object is to give an account of Siegel's formula for $\zeta_K(1-2m)$ for general K, to describe the form it takes when K is quadratic, and prove it in this special case by direct analytic methods. We have tried to keep prerequisites to a minimum by reviewing the main facts about zeta functions of fields (in §1) and the arithmetic of quadratic fields (in §2). We give an exposition of Siegel's theorem and proof in Section 1.

When K is a quadratic field, it is very easy to obtain elementary formulas for $\zeta_K(1-2m)$ directly, using the decomposition $\zeta_K(s) = \zeta(s) L(s, \chi)$. These formulas are discussed in §2. In the simplest case, namely $m = 1$

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and $K = \mathbf{Q}(\sqrt{p})$ with $p \equiv 1 \pmod 4$ a prime number, the formula in question reads

$$\zeta_K(-1) = \frac{1}{24p} \sum_{j=1}^{p-1} \left(\frac{j}{p}\right) j^2, \tag{1}$$

where $\left(\frac{j}{p}\right)$ is the Legendre-Jacobi symbol.

In §3 we return to the Siegel formula and specialize it to the case of real quadratic fields. Because the arithmetic of quadratic fields is completely known and very simple (the different is a principal ideal; the splitting of a rational prime p depends only on the value $+1$, 0, -1 of $\chi(p)$), we can completely evaluate the terms of this formula, arriving at a formula for $\zeta_K(1-2m)$ not involving any notions of algebraic number theory. For instance, in the case above ($m = 1$, discriminant of K a prime p), the formula is

$$\zeta_K(-1) = \frac{1}{30} \sum a\,, \tag{2}$$

where the sum is over all ways of writing $p = b^2 + 4ac$ with a, b and c positive integers. We also discuss bounds for the denominator of $\zeta_K(1-2m)$ (the importance of which was mentioned above) and give tables for $m \leqslant 6$, discriminant of $K \leqslant 50$.

The elementary character of the right-hand sides of (1) and (2) suggests the problem of proving their equality directly, by reasoning involving only finite sums. This is probably impossible: it is not even easy to see *a priori* why the sum in (2) must be divisible by 5 if p is a prime different from 5. However, it is possible to study the sum (2) by the methods of analytic number theory. To do this, we observe that the right-hand side of (2) is the coefficient of $e^{2\pi i p z}$ in the Fourier expansion of a function which is (up to a factor) the product of a theta-function and an Eisenstein series. This function transforms in a known way under the action of the modular group, and therefore one can describe its asymptotic behaviour as z tends towards any rational point on the real axis. This is precisely the sort of problem for which the Hardy-Littlewood circle method was designed. When we apply it, we obtain a "singular series" which approximates (2) and which, on the other hand, can be explicitly summed to yield (1). However, we do not obtain a proof of (2): there is a built-in error in the circle method in this situation, and we cannot show that the singular series really sums to the expression in (2), but only that the error is of smaller order than the main term (roughly the square root) as $p \to \infty$. Indeed, in working out the analogous formula for $\zeta_K(1-2m)$, where $m \geqslant 3$, we find that there really is a difference of this order between the Fourier coefficient we are trying to evaluate and the value of the singular series. The calculation of the singular series is carried out in Section 4.

Finally, in §5 we give conjectures concerning the Fourier coefficients of a certain modular form of weight $4m$ related to the value of $\zeta_K(1-2m)$.

§1. Siegel's Formula

In this section, we will state the formula of Siegel for the value of $\zeta_K(b)$ where K is a totally real algebraic number field and b a negative odd integer. We will also give a brief description of the proof.

We begin by reviewing the main properties of the zeta-function of a field. Let K be an algebraic number field of degree n, and $\mathcal{O}$ the ring of integers in K. For any non-zero ideal $\mathfrak{A}$ of $\mathcal{O}$, the *norm* $N(\mathfrak{A})$ is defined as the number of elements in the quotient $\mathcal{O}/\mathfrak{A}$. For $m = 1, 2, \ldots$, let $i(m)$ denote the number of ideals of $\mathcal{O}$ with norm m. This number is finite for each m and has polynomial growth as $m \to \infty$, and so the series $\Sigma_{m=1}^{\infty} i(m)\, m^{-s}$ makes sense and is convergent if s is a complex number with sufficiently large real part. The function it defines can be extended meromorphically to the whole s-plane, and the function obtained is denoted $\zeta_K(s)$. Thus we have the two representations.

$$\zeta_K(s) = \sum_{\mathfrak{A} \subset \mathcal{O}} \frac{1}{N(\mathfrak{A})^s} \tag{1}$$

$$= \prod_{\mathfrak{P}} (1 - N(\mathfrak{P})^{-1})^{-s}, \tag{2}$$

provided that $Re(s)$ is large enough. The sum in (1) is to be taken over all non-zero ideals of $\mathcal{O}$, and the product in (2) *(Euler product)* over all prime ideals. The function obtained by analytic continuation has a simple pole at $s = 1$ and is holomorphic everywhere else.

Moreover, the function ζ_K satisfies a *functional equation* relating $\zeta_K(s)$ and $\zeta_K(1-s)$. In the case of a totally real field K (i.e. $K = \mathbf{Q}(\alpha)$ where α satisfies a polynomial of degree n with n real roots), this takes the form

$$F(s) = F(1-s), \tag{3}$$

where

$$F(s) = D^{s/2}\, \pi^{-ns/2}\, \Gamma\left(\frac{s}{2}\right)^n \zeta_K(s). \tag{4}$$