# 12. Sums over several intervals of equal length

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*Proof.* The desired congruences follow from Corollaries 7.5, 9.3, and 11.6.

Lerch [44, pp. 409, 410] has derived some class number formulas in terms of the sums $S_{24,i}$, $1 \leqq i \leqq 12$. Karpinski [42] and Rédei [57] have also established class number relations of this sort.

## 12. Sums over several intervals of equal length

In this section, it will be convenient to use the following character analogues of the Poisson summation formula [6, Theorem 2.3], [7, equations (4.1), (4.2)]. Let $f$ be continuous and of bounded variation on $[c, d]$. Let $\chi$ be a primitive character of modulus $k$. If $\chi$ is even, then

$$\text{(12.1)} \quad \sum_{c \leqq n \leqq d}' \chi(n) f(n) = \frac{2G(\chi)}{k} \sum_{n=1}^{\infty} \bar{\chi}(n) \int_c^d f(x) \cos(2\pi n x/k)\, dx\,;$$

if $\chi$ is odd, then

$$\text{(12.2)} \quad \sum_{c \leqq n \leqq d}' \chi(n) f(n) = -\frac{2iG(\chi)}{k} \sum_{n=1}^{\infty} \bar{\chi}(n) \int_c^d f(x) \sin(2\pi n x/k)\, dx\,.$$

The primes $'$ on the summation signs on the left sides of (12.1) and (12.2) indicate that if $c$ or $d$ is an integer, then the associated summands must be halved.

Throughout the section, it is assumed that $\chi$ is a primitive character of modulus $k$. For each of the theorems below, deductions concerning the signs of the pertinent character sums are trivial. Likewise, the corresponding formulas for class numbers are immediate from (2.4). Thus, none of these obvious corollaries shall be explicitly stated.

THEOREM 12.1. Let $\chi$ be even, and let $m$ be any positive integer. Then

$$\text{(12.3)} \quad S_{4m,1} + S_{4m,4} + S_{4m,5} + S_{4m,8} + S_{4m,9} + \ldots + S_{4m,4m} = \frac{2G(\chi)}{\pi} \bar{\chi}(m)\, L(1, \bar{\chi}_{4k})\,.$$

*Proof.* Apply (12.1) several times with $f(x) \equiv 1$ in each case and with $(c, d) = (0, k/4m)$, $(3k/4m, 5k/4m)$, $(7k/4m, 9k/4m)$, ..., $((4m-1)\,k/4m, k)$. We then get

$$S_{4m,1} = \frac{G(\chi)}{\pi} \sum_{n=1}^{\infty} \frac{\bar{\chi}(n)}{n} \sin(2\pi n/4m),$$

$$S_{4m,4} + S_{4m,5} = \frac{G(\chi)}{\pi} \sum_{n=1}^{\infty} \frac{\bar{\chi}(n)}{n} \{ \sin(10\pi n/4m) - \sin(6\pi n/4m) \}$$

$$\vdots$$

$$S_{4m,4m} = \frac{G(\chi)}{\pi} \sum_{n=1}^{\infty} \frac{\bar{\chi}(n)}{n} \{ - \sin(2\pi n(4m-1)/4m) \}.$$

Adding the above equations, we find that

$$(12.4) \qquad S_{4m,1} + S_{4m,4} + S_{4m,5} + \dots + S_{4m,4m} = \frac{G(\chi)}{\pi} \sum_{n=1}^{\infty} \frac{\bar{\chi}(n)}{n} \sum_{j=0}^{2m-1} (-1)^j \sin\big(2(2j+1)\pi n/4m\big).$$

Now an elementary calculation shows that

$$(12.5) \qquad \sum_{j=0}^{2m-1} (-1)^j \sin\big(2(2j+1)\pi n/4m\big) = \begin{cases} 2m(-1)^{\mu}, & \text{if } n = (2\mu+1)m, \\ 0, & \text{otherwise.} \end{cases}$$

Putting (12.5) into (12.4), we conclude that

$$S_{4m,1} + S_{4m,4} + S_{4m,5} + \dots + S_{4m,4m} = \frac{2G(\chi)}{\pi} \sum_{\mu=0}^{\infty} \frac{(-1)^{\mu} \bar{\chi}((2\mu+1)m)}{2\mu+1} = \frac{2G(\chi)}{\pi} \bar{\chi}(m) L(1, \bar{\chi}_{4k}),$$

which completes the proof.

Observe that if $m = 1$, Theorem 12.1 reduces to Theorem 3.7. If $m = 2, 3, 4$, and $6$, then Theorem 12.1 reduces to results that can be derived from Theorems 7.1, 9.1, 10.1, and 11.1, respectively.

THEOREM 12.2. *Let $\chi$ be odd, and let $m$ be a positive integer. If $m$ is odd, then*

$$(12.6) \qquad \sum_{1 \leq j \leq m/2} \left(\frac{m+1}{2} - j\right) S_{m,j} = \frac{G(\chi)}{2\pi i} \{ m - \bar{\chi}(m) \} L(1, \bar{\chi});$$

*if $m$ is even, then*

$$(12.7) \qquad \sum_{1 \leq j \leq m/2} \left(\frac{m+2}{2} - j\right) S_{m,j} = \frac{G(\chi)}{2\pi i} \{ m + 2 - \bar{\chi}(2) - \bar{\chi}(m) \} L(1, \bar{\chi}).$$

*Proof.* Apply (12.2) several times with $f(x) \equiv 1$ in each case and with $(c, d) = (0, k/m), (k/m, 2k/m), \ldots, (([m/2]-1)\,k/m, [m/2]\,k/m)$. We then get

$$S_{m,1} = \frac{iG(\chi)}{\pi} \sum_{n=1}^{\infty} \frac{\bar{\chi}(n)}{n} \{ \cos(2\pi n/m) - 1 \},$$

$$S_{m,2} = \frac{iG(\chi)}{\pi} \sum_{n=1}^{\infty} \frac{\bar{\chi}(n)}{n} \{ \cos(4\pi n/m) - \cos(2\pi n/m) \},$$

$$\vdots$$

$$S_{m,[m/2]} = \frac{iG(\chi)}{\pi} \sum_{n=1}^{\infty} \frac{\bar{\chi}(n)}{n} \{ \cos(2[m/2]\,\pi n/m) - \cos(2\{[m/2]-1\}\,\pi n/m) \}.$$

Multiply the $j$-th equation above by $[m/2]+1-j$, $1 \leqq j \leqq [m/2]$, and add the resulting equations to obtain

$$\sum_{1 \leqq j \leqq m/2} \{[m/2]+1-j\}\, S_{m,j} = \frac{iG(\chi)}{\pi} \sum_{n=1}^{\infty} \frac{\bar{\chi}(n)}{n} \left\{ -[m/2] + \sum_{j=1}^{[m/2]} \cos(2\pi n j/m) \right\}. \tag{12.8}$$

First, suppose that $m$ is odd. Then (12.8) becomes

$$\sum_{1 \leqq j \leqq m/2} \left( \frac{m+1}{2} - j \right) S_{m,j} = \frac{iG(\chi)}{2\pi} \sum_{n=1}^{\infty} \frac{\bar{\chi}(n)}{n} \left\{ -m + \sum_{j=0}^{m-1} \cos(2\pi n j/m) \right\}$$
$$= \frac{iG(\chi)}{2\pi} \{ -m + \bar{\chi}(m) \}\, L(1, \bar{\chi}),$$

from which (12.6) follows.

Suppose next that $m$ is even. Then (12.8) becomes

$$\sum_{1 \leqq j \leqq m/2} \left( \frac{m+2}{2} - j \right) S_{m,j}$$
$$= \frac{iG(\chi)}{2\pi} \sum_{n=1}^{\infty} \frac{\bar{\chi}(n)}{n} \left\{ -m-1+(-1)^n + \sum_{j=0}^{m-1} \cos(2\pi n j/m) \right\}$$
$$= \frac{iG(\chi)}{2\pi} \{ -m-1+\bar{\chi}(2)-1+\bar{\chi}(m) \}\, L(1, \bar{\chi}),$$

from which (12.7) follows.

We indicate some special cases of the previous theorem. If $m = 2$, (12.7) reduces to Theorem 3.2. If $m = 3$, (12.6) yields Theorem 4.1. If $m = 5$, 6, 8, 10, 12, and 24 in Theorem 12.2, we obtain results deducible from Theorems 5.1, 6.1, 7.1, 8.1, 9.1 and 11.1, respectively.

THEOREM 12.3. Let $\chi$ be even and let $m$ be an arbitrary positive integer. Then

$$(12.9)\quad S_{8m,1} - S_{8m,4} - S_{8m,5} + S_{8m,8} + S_{8m,9} - - + + \cdots + S_{8m,8m} = \frac{2^{3/2}\, G(\chi)}{\pi}\, \bar{\chi}(m)\, L(1, \bar{\chi}_{8k}).$$

*Proof.* Apply (12.1) several times with $f(x) \equiv 1$ in each instance and with $(c, d) = (0, k/8m)$, $(3k/8m, 5k/8m)$, $(7k/8m, 9k/8m)$, ..., $((8m-1)\,k/8m, k)$. Accordingly, we find that

$$\begin{aligned} S_{8m,1} - S_{8m,4} - S_{8m,5} + S_{8m,8} + S_{8m,9} - - + + \cdots + S_{8m,8m} &= \frac{G(\chi)}{\pi} \sum_{n=1}^{\infty} \frac{\bar{\chi}(n)}{n} \sum_{j=0}^{8m-1} \chi_4(j)\, \chi_8(j)\, \sin(2\pi nj/8m) \\ &= \frac{G(\chi)}{\pi} \sum_{n=1}^{\infty} \frac{\bar{\chi}(n)}{n} \sum_{\nu=0}^{7} \chi_4(\nu)\, \chi_8(\nu) \sum_{\mu=0}^{m-1} \sin(2\pi n(8\mu+\nu)/8m) \\ &= \frac{G(\chi)}{\pi}\, \bar{\chi}(m) \sum_{n=1}^{\infty} \frac{\bar{\chi}(n)}{n} \sum_{\nu=0}^{7} \chi_4(\nu)\, \chi_8(\nu)\, \sin(2\pi n\nu/8). \end{aligned}$$

The inner sum above is merely $-iG(n, \chi_4\chi_8) = \chi_4(n)\, \chi_8(n)\, 2^{3/2}$, by (2.2). Hence, (12.9) immediately follows.

The special cases with $m = 1$, 2 and 3 of Theorem 12.3 may be deduced from Theorems 7.1, 10.1 and 11.1, respectively.

The proofs of the next four theorems are very similar to the preceding proofs and so will not be given.

THEOREM 12.4. Let $\chi$ be odd, and let $m$ be an arbitrary positive integer. Then

$$S_{8m,2} + S_{8m,3} - S_{8m,6} - S_{8m,7} + + - - \cdots - S_{8m,8m-2} - S_{8m,8m-1} = -\frac{i2^{3/2}\, G(\chi)}{\pi}\, \bar{\chi}(m)\, L(1, \bar{\chi}_{8k}).$$

The special cases of Theorem 12.4 with $m = 1$, 2 and 3 are consequences of Theorems 7.1, 10.1 and 11.1, respectively.

THEOREM 12.5. Let $\chi$ be even, and let $m$ be an arbitrary positive integer. Then

$$\sum_{j=0}^{m-1} S_{3m,3j+2} = - \frac{3^{1/2} G(\chi)}{\pi} \bar{\chi}(m) L(1, \bar{\chi}_{3k}) .$$

The instances of Theorem 12.5 with $m = 1, 2, 4$ and $8$ are consequences of Theorems 4.1, 6.1, 9.1 and 11.1, respectively.

THEOREM 12.6. Let $\chi$ be odd, and let $m$ be an arbitrary positive integer. Then

$$S_{5m,2} - S_{5m,4} + S_{5m,7} - S_{5m,9} + - \cdots + S_{5m,5m-3} - S_{5m,5m-1}$$
$$= - \frac{i5^{1/2} G(\chi)}{\pi} \bar{\chi}(m) L(1, \bar{\chi}_{5k}) .$$

The special cases of Theorem 12.6 for $m = 1$ and $m = 2$ follow immediately from Theorems 5.1 and 8.1, respectively.

THEOREM 12.7. Let $\chi$ be odd, and let $m$ be an arbitrary positive integer. Then

$$S_{12m,2} + S_{12m,3} + S_{12m,4} + S_{12m,5} - S_{12m,8} - S_{12m,9} - S_{12m,10} - S_{12m,11}$$
$$+ + + + - - - - \cdots - S_{12m,12m-4} - S_{12m,12m-3} - S_{12m,12m-2} - S_{12m,12m-1}$$
$$= - \frac{i(12)^{1/2} G(\chi)}{\pi} \bar{\chi}(m) L(1, \bar{\chi}_{12k}) .$$

The special instances of $m = 1$ and $m = 2$ of Theorem 12.7 yield results that are easily deduced from Theorems 9.1 and 11.1, respectively.

The class number formula arising from Theorem 12.1 was first proved by Holden [39]. A less general form of Theorem 12.2 was also established by Holden [36] who in another paper [37] used his result to derive formulas for sums of the Legendre-Jacobi symbol over various residue classes. The special case $m = 1$ of the class number formula deducible from Theorem 12.7 is due to Lerch [44, p. 407]. Otherwise, the results of this section appear to be new.

## 13. SUMS OF QUADRATIC RESIDUES AND NONRESIDUES

We mentioned in the Introduction the two equivalent formulations of Dirichlet's theorem for primes that are congruent to 3 modulo 4. In this section, we state and prove as many theorems as we can that are of the same