2. A TAUBERIAN THEOREM

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CHANGES OF SIGN OF $\pi(x) - \operatorname{li} x$

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1. INTRODUCTION

The prime number theorem asserts that $\pi(x)$, the number of primes not exceeding x, is asymptotic to

$$\operatorname{li} x \overset{def}{=} \lim_{\varepsilon \to 0+} \left(\int_0^{1-\varepsilon} + \int_{1+\varepsilon}^{x} \right) \frac{du}{\log u}$$

as $x \to \infty$. It has been shown [12, p. 72] that $\pi(x) < \operatorname{li} x$ for $3/2 \leqslant x \leqslant 10^8$, and it was once conjectured that this inequality prevailed for all $x \geqslant 3/2$. However, this conjecture was disproved by Littlewood [11] who established

THEOREM 1. $\pi(x) - \operatorname{li} x$ *changes sign infinitely often as* $x \to \infty$.

Littlewood's proof was simplified by Ingham [5]. In the present article we make a further simplification by eliminating use of the so called explicit formula for ψ (cf. [4], pp. 76-80). The deepest fact which we require from analytic number theory is an estimate of the size of $N(T)$, the number of zeros ρ of the Riemann zeta function satisfying $0 < \operatorname{Im} \rho \leqslant T$.

The key step in the argument is Theorem 2, which is given in the next section. This result, which is based on another article of Ingham [6], enables us to relate a certain average of the function π to zeros of the Riemann zeta function.

2. A TAUBERIAN THEOREM

We begin by giving an extension of the Wiener-Ikehara tauberian theorem. Our result admits poles and certain "lesser" singularities on the abscissa of convergence of the transformed function. We adhere to the curious convention of expressing the complex variable s as $\sigma + it$.

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THEOREM 2. *Let F be a real valued function on* $[1, \infty)$ *which is continuous from the right and satisfies a one sided bound*

$$F(x) < \log^{\beta} x \quad \textit{or} \quad F(x) > -\log^{\beta} x$$

for some $\beta < 1$ *and all sufficiently large x. Let*

$$G(s) \overset{def}{=} \int_{1}^{\infty} x^{-s-1} F(x)\, dx$$

converge for $\sigma > 0$. *Let* $T > 0$ *and suppose that there exists a function*

$$H(s) \overset{def}{=} \sum_{|\gamma_n| < T} \frac{a_n}{s - i\gamma_n}$$

(for some choice of complex a_n*'s and real* γ_n*'s) such that the family of restricted functions*

$$t \mapsto G(\sigma + it) - H(\sigma + it) \overset{def}{=} J_{\sigma}(t) \qquad (-T \leqslant t \leqslant T)$$

is Cauchy in L^1 *norm as* $\sigma \to 0+$, *i. e.*

$$\lim_{\sigma,\, \sigma' \to 0+} \int_{-T}^{T} | J_{\sigma}(t) - J_{\sigma'}(t) |\, dt = 0\,.$$

Then, as $y \to \infty$

$$\int_{u=1}^{\infty} F(u)\, K_T(y - \log u) \frac{du}{u} = \sum_{|\gamma_n| < T} a_n \left(1 - \frac{|\gamma_n|}{T}\right) e^{i\gamma_n y} + o_T(1)\,.$$

We have set

$$K_T(x) = \frac{1}{2\pi} \int_{-T}^{T} \left(1 - \frac{|t|}{T}\right) e^{ixt}\, dt \qquad (-\infty < x < \infty)\,.$$

This is the so called Fejér kernel. Also, $o_T(1)$ denotes a function of y and T which, for fixed T, tends to zero as $y \to \infty$.

Proof of Theorem 2. The general strategy is to multiply the equation defining G by $(1 - |t|/T)\, e^{ity}\, dt$, integrate, and evaluate the resulting formula as $\sigma \to 0+$ and then let $y \to \infty$.

For $\sigma > 0$ we have

$$\int_{-T}^{T}\left\{\int_{1}^{\infty} u^{-s-1} F(u)\,du\right\}\left(1-\frac{|t|}{T}\right) e^{ity}\,dt =$$

(1)

$$\int_{-T}^{T} J_\sigma(t)\left(1-\frac{|t|}{T}\right) e^{ity}\,dt + \int_{-T}^{T}\left(1-\frac{|t|}{T}\right) e^{ity} \sum_{|\gamma_n|<T} \frac{a_n}{s-i\gamma_n}\,dt\,.$$

We treat the last integral first. For γ real, $|\gamma| < T$, $\sigma > 0$, $y > 0$, we estimate

$$\int_{-T}^{T}\left(1-\frac{|t|}{T}\right) e^{ity}(s-i\gamma)^{-1}\,dt =$$

$$\left(1-\frac{|\gamma|}{T}\right)\int_{-T}^{T}\frac{e^{ity}}{s-i\gamma}\,dt + \frac{1}{T}\int_{-T}^{T}\frac{|\gamma|-|t|}{s-i\gamma}\,e^{ity}\,dt$$

$$= \mathrm{I} + \mathrm{II}, \text{ say.}$$

We treat I using complex integration:

$$\int_{-T}^{T}\frac{e^{ity}}{s-i\gamma}\,dt = \frac{e^{y(i\gamma-\sigma)}}{i}\int_{z=y(\sigma-iT-i\gamma)}^{y(\sigma+iT-i\gamma)}\frac{e^z\,dz}{z}$$

$$= 2\pi e^{-\sigma y+i\gamma y} + O\left\{\frac{1}{y(T-|\gamma|)}\right\}.$$

The last formula results from replacing the integral over the line segment by one over the other three sides of a rectangle with vertices

$$y(\sigma \pm iT - i\gamma), \qquad y(-m \pm iT - i\gamma),$$

applying the residue theorem, and letting $m \to +\infty$. The constant implied by the O is absolute. Thus

$$\lim_{\sigma\to 0+} I = 2\pi\left(1-\frac{|\gamma|}{T}\right) e^{i\gamma y} + O\left(\frac{1}{yT}\right).$$

We estimate II after noting that $\dfrac{|\gamma|-|t|}{\sigma+i(t-\gamma)}$ is bounded by 1 in absolute value and converges to $(|\gamma|-|t|)/(it-i\gamma)$ for $t \neq \gamma$. Thus

$$\lim_{\sigma\to 0+}\int_{-T}^{T}\frac{|\gamma|-|t|}{s-i\gamma}\,e^{ity}\,dt = \int_{-T}^{T}\frac{|\gamma|-|t|}{i(t-\gamma)}\,e^{ity}\,dt$$

by the dominated convergence theorem. The last expression tends to zero as $y \to \infty$ by the Riemann-Lebesgue lemma, which asserts that the Fourier transform of an L^1 function vanishes "at infinity" (cf. [7], p. 123).

It follows that

$$\lim_{\sigma \to 0+} \int_{-T}^{T} \left(1 - \frac{|t|}{T}\right) e^{ity} \sum_{|\gamma_n| < T} \frac{a_n}{s - i\gamma_n} \, dt$$

$$= 2\pi \sum_{|\gamma_n| < T} \left(1 - \frac{|\gamma_n|}{T}\right) a_n e^{i\gamma_n y} + \varphi(y),$$

where $\varphi(y) \to 0$ as $y \to \infty$.

The function J_σ in (1) converges in L^1 norm to a function J in $L^1[-T, T]$ by use of the Cauchy hypothesis and the fact that L^1 is complete. Thus

$$\lim_{\sigma \to 0+} \int_{-T}^{T} J_\sigma(t) \left(1 - \frac{|t|}{T}\right) e^{ity} \, dt = \int_{-T}^{T} J(t) \left(1 - \frac{|t|}{T}\right) e^{ity} \, dt,$$

and the last integral tends to zero as $y \to +\infty$ by another application of the Riemann-Lebesgue lemma. (The completeness of L^1 could be avoided by use of the argument in [1, pp. 190-194].)

The left side of (1) equals

$$\int_{-T}^{T} \left\{ \int_{1}^{\infty} u^{-\sigma-1} F(u) e^{it(y - \log u)} \, du \right\} \left(1 - \frac{|t|}{T}\right) dt$$

$$= \int_{1}^{\infty} u^{-\sigma-1} F(u) \left\{ \int_{-T}^{T} \left(1 - \frac{|t|}{T}\right) e^{it(y - \log u)} \, dt \right\} du$$

$$= 2\pi \int_{1}^{\infty} u^{-\sigma-1} F(u) K_T(y - \log u) \, du.$$

The interchange of integrations is justified by the fact that the integral for G converges uniformly on the line segment $\{s = \sigma + it\colon -T \leqslant t \leqslant T\}$ for any fixed positive numbers σ and T. Integration shows that

$$\text{(2)} \qquad K_T(x) \overset{def}{=} \frac{1}{2\pi} \int_{-T}^{T} \left(1 - \frac{|t|}{T}\right) e^{ixt} \, dt = \frac{T}{2\pi} \left(\frac{\sin \, Tx/2}{Tx/2}\right)^2$$

and hence for any $\beta < 1$

$$\int_1^\infty u^{-1} \log^\beta u \, K_T(y - \log u)\, du < \infty .$$

We write

$$F(u) = \left(F(u) - K \log^\beta u\right) + K \log^\beta u ,$$

and note that $F(u) - K \log^\beta u$ is of one sign for suitable K and some $\beta < 1$. By the monotone convergence theorem

$$\int_1^\infty u^{-\sigma-1} F(u)\, K_T(y - \log u)\, du \longrightarrow$$

$$\int_1^\infty u^{-1} \left(F(u) - K \log^\beta u\right) K_T(y - \log u)\, du$$

$$+ \int_1^\infty u^{-1} K \log^\beta u \, K_T(y - \log u)\, du$$

as $\sigma \to 0+$. The last integral has just been shown finite. The integral involving $F(u) - K \log^\beta u$ must also be finite, since each of the other terms in (1) has a finite limit as $\sigma \to 0+$.

Assembling the estimates of the terms in (1), we obtain the desired formula. #

In particular, if $F(u) = 1$ for $1 \leqslant u < \infty$, then $G(s) = \int_1^\infty x^{-s-1}\, dx = 1/s$ and the theorem yields

$$\int_{u=1}^\infty K_T(y - \log u)\, u^{-1}\, du = 1 + o(1)$$

or

$$\int_{-\infty}^\infty K_T(v)\, dv = 1 .$$