# Introduction

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **17 (1971)**

Heft 1: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **23.09.2024**

# ON THE EXISTENCE AND THE REGULARITY OF SOLUTIONS OF LINEAR PSEUDO-DIFFERENTIAL EQUATIONS [1]

by Lars HÖRMANDER

## INTRODUCTION

Let $X$ be an open set in $\mathbf{R}^n$ (or a $C^\infty$ manifold) and let $P$ be a differential operator in $X$ with $C^\infty$ coefficients. Our purpose is to study the equation

$$Pu = f$$

where $u$ and $f$ are functions or distributions in $X$. Somewhat vaguely we can state the questions to be considered as follows:

*a)* What are the conditions on $P$ and on $X$ for a local or a global existence theorem to be valid ?

*b)* What are the relations between the singularities of $u$ and those of $f$ when $X$ and $P$ are given ?

In fact, these questions are so closely related that they can be considered as different forms of the same problem.

We shall look for answers in terms of geometric properties of the characteristics. These are defined as follows. If $P$ is of order $m$ and $u$, $\varphi \in C^\infty(X)$, then

$$P(e^{i\omega\varphi}u) = e^{i\omega\varphi}\left(\omega^m p(x, \operatorname{grad}\varphi)\, u + \omega^{m-1} Lu + \ldots\right)$$

where $p$ is a homogeneous function of degree $m$ on the cotangent bundle $T^*(X)$, called the characteristic polynomial or principal part (symbol) of $P$, and $L$ is a first order differential operator depending on $\varphi$. The zeros of $p$ in $T^*(X)\setminus 0$ are called (real) characteristics. The Hamilton-Jacobi integration theory for the characteristic equation $p(x, \operatorname{grad}\varphi) = 0$ also introduces certain curves in the level surfaces of $\varphi$, the bicharacteristics (see section 3.1). The classical methods used to relate geometrical and wave

---

[1] Survey lectures given by invitation at the Institute for Advanced Study on March 22, 25, 29, April 1, 5 and 8, 1971, under the sponsorship of the International Mathematical Union, while the author was supported in part by National Science Foundation grant GP-7952X2.

optics (or classical and wave mechanics) consist in first choosing a solution of the equation $p(x, \operatorname{grad} \varphi) = 0$ and then determining $u$ as a solution of the (transport) equation $Lu = 0$, which is a first order equation along the bicharacteristics. This makes $P(e^{i\omega\varphi} u)$ of order $\omega^{m-2}$. There is also a much more refined method due to Luneburg (see Kline-Kay [1]), in which $u$ is replaced by an asymptotic series $\sum_{0}^{\infty} u_j \omega^{-j}$ in $\omega$. By successive choice of the functions $u_j$ one can obtain an asymptotic solution of the equation $P(ue^{i\omega\varphi}) = 0$. The importance of these expansions for the general theory of partial differential equations has been realized for quite some time (see Lax [1]) but it is only during the last few years that one has started to exploit them systematically. An important step in this direction was taken by Egorov [1] who also called attention to the results of Maslov [1]. A rigorous and systematic exposition incorporating the theory of pseudo-differential operators has been undertaken in Hörmander [6, 9, 10] and we shall discuss some of the results in Chapter II. Related ideas will be applied to differential equations with constant coefficients in Chapter I.

When discussing singularities we shall not only be concerned with their location but also with their local harmonic analysis. For a distribution in $X$ this leads to a set in the cosphere bundle of $X$. In the case of hyperfunctions the advantages of such a point of view were first pointed out by Sato [1]. Many variations of this idea are possible, and we shall indicate some (sections 1.6 and 2.2). The sets obtained by harmonic analysis of the singularities will be called wave front sets here. Since the geometrical objects associated with $P$, such as the characteristics, usually live in the cotangent bundle of $X$, it is natural to expect that they can be more easily related to the wave front set than to the set of singularities.

Since much work is in progress on the topics discussed in these lectures it seems useless to try to give a complete picture of the present state of the theory, but we do attempt to indicate the most important directions of this work. For references to some topics not discussed at all here see also Hörmander [13]. Originally we had planned to include a number of results concerning operators with variable coefficients involving the principal and the subprincipal part, that is, an invariantly defined function of degree one less than the principal part. However, these were left out because of the already excessive length of the survey, and we content ourselves with referring to Hörmander [5] (second order hypoelliptic differential equations), Radkevič [1], [2] (simplifications and extensions of these results), a forthcoming book on hypoelliptic operators by O. A. Olejnik and E. V. Radkevič; Mizohata-Ohya [1] and Flaschka-Strang [1] (hyperbolic operators with characteristics of constant multiplicity). The methods discussed in Chapter III can obviously be used to push much further in this direction. For the constant coefficient case a model result is given by Theorem 1.5.1.

# Chapter I

## OPERATORS WITH CONSTANT COEFFICIENTS

### 1.1. *Fundamental solutions*

A differential operator with constant coefficients in $\mathbf{R}^n$ can be written in the form $P(D)$ where $P$ is a polynomial in $n$ variables with complex coefficients and $D = (-i\partial/\partial x_1, \ldots, -i\partial/\partial x_n)$. Explicitly

$$P(D) = \sum a_\alpha D^\alpha$$

where $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a multi-index and the sum is finite.

It is easy to show that the equation

$$P(D)u = f \tag{1.1.1}$$

can always be solved locally. To do so we assume first that $f \in C_0^\infty$. If $u$ is a solution of (1.1.1) with a well defined Fourier transform $\hat{u}$, we must have $P(\xi)\,\hat{u}(\xi) = \hat{f}(\xi)$, and so by Fourier's inversion formula

$$u(x) = (2\pi)^{-n} \int e^{i<x,\xi>} \hat{f}(\xi)/P(\xi)\, d\xi\,. \tag{1.1.2}$$

However, $P$ may have zeros in or near $\mathbf{R}^n$ and this makes it necessary to deform the integration contour in order to obtain a well defined solution from (1.1.2).

First note that if $\Phi \in C_0^\infty(\mathbf{C}^n)$ and

$$\Phi(e^{i\theta}\zeta) = \Phi(\zeta)\,,\quad \theta \in \mathbf{R}\,,\ \int \Phi(\zeta)\, d\lambda(\zeta) = 1\,, \tag{1.1.3}$$

where $d\lambda$ is the Lebesgue measure in $\mathbf{C}^n$, then

$$\int F(\zeta)\, \Phi(\zeta)\, d\lambda(\zeta) = F(0) \tag{1.1.4}$$

for any entire analytic function $F$. In fact, by Cauchy's integral formula

$$\int F(\zeta e^{i\theta})\, d\theta = 2\pi F(0)\,,$$

and if we multiply by $\Phi(\zeta)$ and integrate, (1.1.3) gives (1.1.4).