§ 8. Discussion of case (i) : G not 0-dimensional

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$$u_n = n^{-2} Q_{j_n}$$

satisfy the conditions

$$\left.\begin{aligned} &\mathrm{sp}\,(u_n) \subseteq \Gamma_0, \sum_{n=1}^{\infty} \| u_n \| < \infty \\ &S_{\Delta_{j_n}} u_n (0) \text{ is real and } > n. \end{aligned}\right\} \tag{7.9}$$

At this point the construction in § 2 will yield integers $0 < n_1 < n_2 < \ldots$ and specifiable sequences $(\gamma_p)_{p \in N}$ of positive numbers such that each function of the form

$$f = \sum_{p=1}^{\infty} \gamma_p u_{n_p}$$

is continuous and satisfies

$$sp\,(f) \subseteq \Gamma_0, \lim_{p \to \infty} \mathrm{Re}\, S_{\Delta_{j_{n_p}}} f(0) = \infty. \tag{7.10}$$

A fortiori, f satisfies (7.3).

We add here that, if the Δ_j are symmetric, the D_{Δ_j} are real-valued, and we may work throughout with real-valued functions, replacing $\mathrm{Re}\, S_{\Delta_j} f$ by $S_{\Delta_j} f$ everywhere.

§ 8. *Discussion of case (i) :* G *not* 0*-dimensional*

8.1 In this case $\Phi \neq \Gamma$, and we begin by considering a finite subset of Γ of the form

$$\Delta = \Omega + \Lambda, \tag{8.1}$$

where Ω and Λ are finite subsets of Γ such that $\pi \mid \Omega$ is 1-1 and $\varnothing \neq \Lambda \subseteq \Phi$. We aim to show that (for a suitable absolute constant $k > 0$)

$$\| D_\Delta \|_1 \geqq k \left(\frac{\log N}{\log \log N} \right)^{\frac{1}{4}}, \tag{8.2}$$

provided $N = \mid \Omega \mid$ (the cardinal number of Ω) is sufficiently large.

8.2 Proof of (8.2). Introduce H as the annihilator in G of Φ and identify in the usual way the dual of H with Γ / Φ. Likewise identify the dual of $K = G/H$ with Φ ([7], (24.11)).

We then have

$$\| D_\Delta \|_1 = \int_G \Big| \sum_{\gamma \in \Delta} \gamma \Big| \, d\lambda_G$$
$$= \int_{G/H} d\lambda_{G/H}(\bar{x}) \int_H \Big| \sum_{\theta \in \Omega} \sum_{\phi \in \Lambda} \theta\,(x+y)\,\phi\,(x+y) \Big| \, d\lambda_H(y),$$

the inner integral being viewed as a function of $\bar{x} = x+H$ Thus, writing $\bar{\theta}$ for $\pi\,(\theta)$ and noting that $\phi\,(y) = 1$ for $\phi \in \Lambda \subseteq \Phi$ and $y \in H$, we obtain

$$\| D_\Delta \|_1 = \int_{G/H} d\lambda_{G/H}(\bar{x}) \int_H \Big| \sum_{\theta \in \Omega} \alpha\,(\theta, \mathrm{x})\bar{\theta}\,(y) \Big| \, d\lambda_H(y), \tag{8.3}$$

where

$$\alpha\,(\theta, x) = \theta\,(x) \sum_{\phi \in \Lambda} \phi\,(x).$$

Now, since the dual of H (namely Γ/Φ) is torsion-free ([7], (A.4)), Theorem A of [8] shows that (for a suitable absolute constant $k > 0$) we have

$$\int_H \Big| \sum_{\theta \in \Omega} \alpha\,(\theta, x)\bar{\theta}\,(y) \Big| \, d\lambda_H(y) \geqq k \left(\frac{\log N}{\log \log N} \right)^{\frac{1}{4}} \min_{\theta \in \Omega} | \alpha\,(\theta, x) |$$
$$= k \left(\frac{\log N}{\log \log N} \right)^{\frac{1}{4}} \Big| \sum_{\phi \in \Lambda} \phi\,(\bar{x}) \Big|, \tag{8.4}$$

since $|\theta\,(x)| = 1$ and $\phi\,(x)$ depends only $\bar{x}$. By (8.3) and (8.4),

$$\| D_\Delta \|_1 \geqq k \left(\frac{\log N}{\log \log N} \right)^{\frac{1}{4}} \int_{G/H} \Big| \sum_{\phi \in \Lambda} \phi\,(\bar{x}) \Big| \, d\lambda_{G/H}(\bar{x}). \tag{8.5}$$

Since $\Lambda \neq \varnothing$, the remaining integral is not less than the maximum modulus of the Fourier transform of the function $\bar{x} \mapsto \sum_{\phi \in \Lambda} \phi\,(\bar{x})$, i.e., is not less than unity. Thus, (8.2) follows from (8.5).

8.3 Proof of 7.4 (i). The conclusions stated in case (i) of 7.4 are now almost immediate. If $\mathscr{D} = (\Delta_j)_{j \in N}$ is a grouping of infinite type covering Γ_0, $|\pi\,(\Delta_j)| \to \infty$ and so, since $\Lambda_j \subseteq \Phi$, $|\pi\,(\Omega_j)| \to \infty$. Then (8.2) shows that (7.6) is satisfied, and it remains only to refer to 7.6.

8.4 Supplementary remarks. The fact that, when G is not 0-dimensional, (7.6) holds for suitable subgroups Γ_0 of Γ and suitable groupings $\mathscr{D} = (\Delta_j)_{j \in N}$ covering Γ_0 can be derived without appeal to Theorem A of [8]. To do this, it suffices to take $\gamma_k \in \Gamma \setminus \Phi$ $(k = 1, 2, \ldots, m)$ such that the family $(\gamma_k)_{1 \leqq k \leqq m}$ is independent (see [7], (A.10)), define

$$\Gamma_0 = \{\textstyle\sum_{k=1}^{m} n_k \gamma_k : n_k \in Z \text{ for } k = 1, 2, \ldots, m\},$$

and make use of the formula

$$\int_G F(\gamma_1(x), \ldots, \gamma_m(x)) d\gamma_G(x)$$
$$= (2\pi)^{-m} \int_0^{2\pi} \ldots \int_0^{2\pi} F(e^{it}, \ldots, e^{it_m}) dt_1 \ldots dt_m, \tag{8.6}$$

valid for every $F \in C(T^m)$, where T denotes the circle group. (Recall that $\sum_{k=1}^{m} n_k \gamma_k$ denotes the character $x \mapsto \gamma_1(x)^{n_1} \ldots \gamma_m(x)^{n_m}$ of G.) It then appears that (7.6) holds when one takes

$$\Delta_j = \{\textstyle\sum_{k=1}^{m} n_k \gamma_k : |n_k| \leqq r_{j,k} \text{ for } k = 1, 2, \ldots, m\},$$

where the $r_{j,k}$ are positive integers satisfying $r_{j,k} < r_{j,k+1}$ and $\lim_{j \to \infty} r_{j,k} = \infty$. Moreover, when $m = 1$, the Cohen-Davenport result (essentially Theorem A of [8] for the case $G = T$) shows that (7.6) holds for every grouping $\mathscr{D}$ covering Γ_0.

The verification of (8.6) is simple. First note that, if G and G' are compact groups, and if ϕ is a continuous homomorphism of G into G', then

$$\int_G (F \circ \phi) d\lambda_G = \int F d\lambda_{\phi(G)} \tag{8.7}$$

for every $F \in C(G')$. (This is a consequence of the fact that $F \mapsto \int_G (F \circ \phi) d\lambda_G$ is invariant under translation by elements of $\phi(G)$, combined with the uniqueness of the normalised Haar measure on a compact group.) Taking $G' = T^m$ and $\phi : x \mapsto (\gamma_1(x), \ldots, \gamma_m(x))$, the stated conditions on the γ_k are just adequate to ensure that the annihilator in Z^m (identified in the canonical fashion with the dual of T^m) of $\phi(G)$ is $\{(0, \ldots, 0)\}$ and so ([7], (24.10)) that $\phi(G) = T^m$. Accordingly, (8.6) appears as a special case of (8.7).

It is perhaps worth indicating that special cases of (8.7) can be exploited in other ways. For example, suppose more generally that κ is an arbitrary nonvoid set and that $(\gamma_k)_{k \in \kappa}$ is a finite or infinite independent family of elements of $\Gamma \setminus \Phi$. Denote by Γ_0 the subgroup of Γ generated by $\{\gamma_k : k \in \kappa\}$. Taking $G' = T^\kappa$ and $\phi : x \mapsto (\gamma_k(x))_{k \in \kappa}$, one may use (8.7) in a similar fashion to show that there is an isometric isomorphism $F \leftrightarrow F \circ \phi = f$ between $L^p(T^\kappa)$ (or $C(T^\kappa)$) and the subspace of $L^p(G)$ (or $C(G)$) formed of those $f \in L^p(G)$ or $C(G)$) such that $\mathrm{sp}(f) \subseteq \Gamma_0$. Moreover, if one identifies in the canonical fashion the dual of T^κ with the weak direct product Z^{κ^*}, the said isomorphism is such that $\hat{F} = \hat{f} \circ \phi'$, where ϕ' is the isomorphism of Z^{κ^*} onto Γ_0 defined by $(n_k) \rightarrow \sum_{k \in \kappa} n_k \gamma_k$.

One consequence of this may be expressed roughly as follows: If the compact Abelian group G is such that $\Gamma \setminus \Phi$ contains an independent family of (finite or infinite) cardinality m, then Fourier series on G behave, in respect of convergence or summability, no better than do Fourier series on T^m.

Another consequence is that, if Δ is a subset of Γ_0, then Δ is a Sidon (or $\Lambda(p)$) subset of Γ if and only if $\phi'^{-1}(\Delta)$ is a Sidon (or $\Lambda(p)$) subset of Z^{κ^*}.

8.5 Further results. Theorem A of [8] implies something stronger than (8.2), namely: if ω is any complex-valued function on Γ such that

$$\omega(\gamma + \phi) = \omega(\gamma) \quad (\gamma \in \Gamma, \phi \in \Phi), \tag{8.8}$$

so that ω can be regarded as a function on Γ/Φ, and if we write

$$D_\Delta^\omega = \sum_{\gamma \in \Delta} \omega(\gamma)\, \bar{\gamma}, \; S_\Delta^\omega f = \sum_{\gamma \in \Delta} \omega(\gamma)\, \hat{f}(\gamma), \tag{8.9}$$

then, for $\Delta = \Omega + \Lambda$ as in (8.1), we have

$$\| D_\Delta^\omega \|_1 \geqq k \left(\frac{\log N}{\log \log N} \right)^{\frac{1}{4}} \min_{\gamma \in \Omega} | \omega(\gamma) | \tag{8.10}$$

provided $N = | \Omega |$ is sufficiently large.

So, if we can arrange for $\Omega = \Omega_j$ to vary in such a way that the right-hand side of (8.10) tends to infinity with j, the substance of 7.6 will lead to a continuous f satisfying $\mathrm{sp}(f) \subseteq \Gamma_0$ and

$$\overline{\lim_{j \to \infty}} \operatorname{Re} S_{\Delta_j}^\omega f(0) = \infty. \tag{8.11}$$

Taking the most familiar case, in which $G = T$, $\Gamma = Z$ and $\Phi = \{0\}$, and supposing $\Delta = \Omega$ to range over a sequence (Δ_j) of finite subsets of Z such that, if $N_j = | \Delta_j |$,

$$\lim_j \left(\frac{\log N_j}{\log \log N_j} \right)^{\frac{1}{4}} \min_{n \in \Delta_j} | \omega(n) | = \infty,$$

the construction will lead to a continuous f on T such that

$$\overline{\lim_j} \operatorname{Re} S_{\Delta_j}^\omega f(0) = \infty.$$

In particular, taking $\Delta_j = \{n \in Z : 2^j \leqq n < 2^{j+1}\}$ it can be arranged that

$$\sum_{n \in Z} \frac{\pm \hat{f}(n)}{(\log(2+|n|))^{\alpha}}$$

diverges for any preassigned distribution of signs $\pm$ and any preassigned $\alpha < \frac{1}{4}$.

Of course, much stronger results are derivable by using random (and unspecifiable!) changes of sign, but there seems little hope of making this even remotely constructive.

§ 9. *Discussion of case (*ii*) :* G 0-*dimensional*

9.1 In this case there is ([7], (7.7)) a base of neighbourhoods of zero in G formed of compact open subgroups W. For each such W the annihilator $\Delta = W^\circ$ in Γ of W is a finite subgroup of Γ. Define

$$k_W = \lambda_G(W)^{-1} \times \text{characteristic function of } W. \tag{9.1}$$

Then k_W is continuous, $k_W \geqq 0$, $\int_G k_W \, d\lambda_G = 1$. The transform $\hat{k}_W$ of k_W is plainly equal to unity on Δ. On the other hand, since W is a subgroup, we have for $a \in W$ and $\gamma \in \Gamma$

$$\begin{aligned} \hat{k}_W(\gamma) &= \int_G k_W(x) \overline{\gamma(x)} \, d\lambda_G(x) = \int_G k_W(x+a) \overline{\gamma(x)} \, d\lambda_G(x) \\ &= \int_G k_W(y) \overline{\gamma(y-a)} \, d\lambda_G(y) \\ &= \gamma(a) \hat{k}_W(\gamma), \end{aligned}$$

which shows that $\hat{k}_W(\gamma) = 0$ if $\gamma \in \Gamma \setminus \Delta$. Thus $\hat{k}_W$ is the characteristic function of Δ, and so

$$k_W = D_{W^\circ}. \tag{9.2}$$

By (9.1) and (9.2), a routine argument shows that, if $1 \leqq p < \infty$ and $f \in L^p(G)$, then

$$f = \lim_W S_{W^\circ} f \tag{9.3}$$

in $L^p(G)$; and that (9.3) holds uniformly for any continuous f.

9.2 Proof of 7.4 (ii). If Γ_0 is any countably infinite subgroup of Γ we can choose a sequence W_j of compact open subgroups of G such that