

6. Application to Tauberian theorems

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$$(5.10) \quad \lim \frac{\mathcal{J}_V(t) - \mathcal{J}_V(\lambda t)}{[\mathcal{J}_U(t) - \mathcal{J}_U(\lambda t)] L(t)} = \\ = \lim \frac{L(t) \mathcal{J}_U(t) - L(\lambda t) \mathcal{J}_U(\lambda t)}{L(t) \mathcal{J}_U(t) - L(\lambda t) \mathcal{J}_U(\lambda t)} = 1.$$

But the fraction on the left lies between

$$\frac{V(\lambda t)}{U(t) L(t)} \quad \text{and} \quad \frac{V(t)}{U(\lambda t) L(t)}$$

and so (5.1) is true.

6. APPLICATION TO TAUBERIAN THEOREMS

If the measure U varies regularly at infinity, then its Laplace transform ω varies regularly at the origin. More precisely, Karamata's now classical Tauberian theorem states that for any $\alpha \geq 0$ and slowly varying function L the two relations

$$(6.1) \quad U(x) \sim x^\alpha L(x) \quad \omega(\lambda) \sim \Gamma(\alpha + 1) \lambda^{-\alpha} L(\lambda^{-1})$$

imply each other; here $x \rightarrow \infty$ but $\lambda \rightarrow 0$. [The sign \sim indicates that the ratio of the two sides tends to 1.] For an example of a probabilistic application suppose that

$$(6.2) \quad U(x) = \int_0^x y^p F(dy)$$

is the truncated p^{th} moment of a probability distribution F on the positive half axis. For simplicity let p stand for a positive integer. Then $U_p(x) = 1 - F(x)$ and $\omega = (-1)^p \phi^{(p)}$ where ϕ is the Laplace-Stieltjes transform of F . If ω varies regularly in accordance with (6.1) then Karamata's relation (1.8) implies that

$$(6.3a) \quad 1 - F(x) \sim \frac{\alpha}{p - \alpha} x^{\alpha - p} L(x) \quad \text{when} \quad \alpha < p$$

$$(6.3b) \quad 1 - F(x) = o(x^\alpha L(x)) \quad \text{when} \quad \alpha = p.$$

(Note that necessarily $0 \leq \alpha \leq p$ because the measure F is finite.) In other words, the behavior at the origin of the derivatives of the Laplace transform determines the behavior of the tail $1 - F(x)$, and vice versa.

From the continuity theorem for Laplace transforms one concludes without difficulties that *dominated variation of U at ∞ is equivalent to dominated variation of ω at 0*. Even in the case of mere dominated variation the behavior of $\omega = (-1)^p \phi^{(p)}$ therefore permits inferences concerning the behavior of the tail $1 - F$, but naturally the conclusions will lack the pleasing precision of (6.3). It is therefore remarkable that precise asymptotic equivalence relations can be obtained when comparing two probability distributions F and G with Laplace transforms ϕ and γ .

A typical *Tauberian ratio limit theorem* would state that the two relations

$$(6.4) \quad \gamma^{(p)}(\lambda) \sim \phi^{(p)}(\lambda) L\left(\frac{1}{\lambda}\right) \quad \lambda \rightarrow 0$$

and

$$(6.5) \quad 1 - G(x) \sim [1 - F(x)] L(x) \quad x \rightarrow \infty$$

imply each other. This is not true in full generality; indeed, (6.3b) points to exceptional situations even when the transforms in (6.4) vary regularly. However, our results yield a variety of fairly general sufficient conditions for the validity of the conclusion. Suppose, for example, that for some constants A and $\alpha < p$

$$(6.6) \quad (-1)^p \phi^{(p)}(\lambda) < A \lambda^{-\alpha}$$

for λ sufficiently small. It is easily seen in this case that U varies dominatedly with exponent α and (6.4) is equivalent to

$$(6.7) \quad V \leftrightarrow UL$$

in the sense of (5.1). (Here V stands for the truncated moment function of G defined as in (6.2).) Theorem 4 then asserts that (5.3) holds, and this implies

$$(6.8) \quad V_p \leftrightarrow U_p L$$

whenever U is bounded away from 0. Now (6.5) differs from (6.8) only notationally, and we know that the condition (4.1) guarantees that R_U is bounded away from 0 and that $U_p = 1 - F$ varies dominatedly. Again, (4.1) holds if, and only if, each limit of a convergent sequence of measures $U(t_n dx)/U(t_n)$ attributes a positive measure to $(0, \infty)$. This requirement is satisfied if, and only if,

$$(6.9) \quad \liminf_{\varepsilon \rightarrow 0} \frac{\phi^{(p)}(\varepsilon \lambda_0)}{\phi^{(p)}(\varepsilon)} < 1$$

for some $\lambda_0 > 1$. Accordingly, if the conditions (6.6) and (6.9) hold then (6.4) implies (6.6.) as well as the dominated variation of $1 - F$ and $1 - G$.

Our results permit various paraphrases of the sufficient conditions, and also of the ratio limit theorem itself. That (6.6) by itself is not sufficient is shown by (6.3b); without (6.9) certain subsequences may exhibit the pattern of slow variation, and the conclusion (6.5) must be replaced by a weaker conclusion of the form (6.3b).

7. ON THE TAILS OF INFINITELY DIVISIBLE DISTRIBUTIONS

To illustrate the usefulness of the notion of dominated variation in probabilistic contexts we prove the following

PROPOSITION. *Let H stand for an infinitely divisible probability distribution with Lévy measure $M \{dx\}$. If M varies dominatedly at $+\infty$ then*

$$(7.1) \quad 1 - H(x) \sim M \{ (x, \infty) \}, \quad x \rightarrow +\infty$$

in the sense that the ratio of the two sides tends to unity at all points of continuity. (A very special case involving regular variation is mentioned in [1], p. 540.)

PROOF. We shall show that the general proposition follows easily from the special case where M is supported by the positive half axis and has a finite mass μ . In this case

$$(7.2) \quad M \{ (x, \infty) \} = \mu [1 - F(x)] \quad x > 0$$

where F is a probability distribution on $(0, x)$, and H reduces to the compound Poisson distribution given by

$$(7.3) \quad H(x) = e^{-\mu} \sum_{n=0}^{\infty} \frac{\mu^n}{n!} F^{n*}(x) \quad x > 0.$$

We proceed to prove the assertion (7.1) for distributions of this form assuming that $1 - F$ varies dominatedly. Note that F^{n*} is the distribution of the sum $S_n = X_1 + \dots + X_n$ of n mutually independent random variables with the common distribution F . Since these variables are positive, the event $\{S_n > x\}$ occurs whenever at least one among the n variables exceeds x , and so

$$(7.4) \quad 1 - F^{n*}(x) \geq n [1 - F(x)] - \binom{n}{2} [1 - F(x)]^2$$