6. Application to Tauberian theorems

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$$(5.10) \qquad \lim \frac{\mathscr{I}_V(t) - \mathscr{I}_V(\lambda t)}{[\mathscr{I}_U(t) - \mathscr{I}_U(\lambda t)]\, L(t)} =$$

$$= \lim \frac{L(t)\, \mathscr{I}_U(t) - L(\lambda t)\, \mathscr{I}_U(\lambda t)}{L(t)\, \mathscr{I}_U(t) - L(t)\, \mathscr{I}_U(\lambda t)} = 1\,.$$

But the fraction on the left lies between

$$\frac{V(\lambda t)}{U(t)\, L(t)} \quad \text{and} \quad \frac{V(t)}{U(\lambda t)\, L(t)}$$

and so (5.1) is true.

6. Application to Tauberian theorems

If the measure U varies regularly at infinity, then its Laplace transform ω varies regularly at the origin. More precisely, Karamata's now classical Tauberian theorem states that for any $\alpha \geqslant 0$ and slowly varying function L the two relations

$$(6.1) \qquad U(x) \sim x^{\alpha} L(x) \qquad \omega(\lambda) \sim \Gamma(\alpha+1)\, \lambda^{-\alpha} L(\lambda^{-1})$$

imply each other; here $x \to \infty$ but $\lambda \to 0$. [The sign $\sim$ indicates that the ratio of the two sides tends to 1.] For an example of a probabilistic application suppose that

$$(6.2) \qquad U(x) = \int_0^x y^p F(dy)$$

is the truncated p^{th} moment of a probability distribution F on the positive half axis. For simplicity let p stand for a positive integer. Then $U_p(x) = 1 - F(x)$ and $\omega = (-1)^p \phi^{(p)}$ where ϕ is the Laplace-Stieltjes transform of F. If ω varies regularly in accordance with (6.1) then Karamata's relation (1.8) implies that

$$(6.3a) \qquad 1 - F(x) \sim \frac{\alpha}{p - \alpha}\, x^{\alpha - p} L(x) \qquad \textit{when} \qquad \alpha < p$$

$$(6.3b) \qquad 1 - F(x) = o(x^{\alpha} L(x)) \qquad \textit{when} \qquad \alpha = p\,.$$

(Note that necessarily $0 \leqslant \alpha \leqslant p$ because the measure F is finite.) In other words, the behavior at the origin of the derivatives of the Laplace transform determines the behavior of the tail $1 - F(x)$, and vice versa.

From the continuity theorem for Laplace transforms one concludes without difficulties that *dominated variation of U at* ∞ *is equivalent to dominated variation of* ω *at* 0. Even in the case of mere dominated variation the behavior of $\omega = (-1)^p \phi^{(p)}$ therefore permits inferences concerning the behavior of the tail $1-F$, but naturally the conclusions will lack the pleasing precision of (6.3). It is therefore remarkable that precise asymptotic equivalence relations can be obtained when comparing two probability distributions F and G with Laplace transforms ϕ and γ.

A typical *Tauberian ratio limit theorem would state that the two relations*

$$\gamma^{(p)}(\lambda) \sim \phi^{(p)}(\lambda)\, L\left(\frac{1}{\lambda}\right) \qquad \lambda \to 0 \tag{6.4}$$

and

$$1 - G(x) \sim [1 - F(x)]\, L(x) \qquad x \to \infty \tag{6.5}$$

imply each other. This is not true in full generality; indeed, (6.3*b*) points to exceptional situations even when the transforms in (6.4) vary regularly. However, our results yield a variety of fairly general sufficient conditions for the validity of the conclusion. Suppose, for example, that for some constants A and $\alpha < p$

$$(-1)^p \phi^{(p)}(\lambda) < A\,\lambda^{-\alpha} \tag{6.6}$$

for λ sufficiently small. It is easily seen in this case that U varies dominatedly with exponent α and (6.4) is equivalent to

$$V \leftrightarrow UL \tag{6.7}$$

in the sense of (5.1). (Here V stands for the truncated moment function of G defined as in (6.2).) Theorem 4 then asserts that (5.3) holds, and this implies

$$V_p \leftrightarrow U_p L \tag{6.8}$$

whenever U is bounded away from 0. Now (6.5) differs from (6.8) only notationally, and we know that the condition (4.1) guarantees that R_U is bounded away from 0 and that $U_p = 1 - F$ varies dominatedly. Again, (4.1) holds if, and only if, each limit of a convergent sequence of measures $U(t_n\, dx)/U(t_n)$ attributes a positive measure to $(0, \infty)$. This requirement is satisfied if, and only if,

$$\liminf_{\varepsilon \to 0} \frac{\phi^{(p)}(\varepsilon\lambda_0)}{\phi^{(p)}(\varepsilon)} < 1 \tag{6.9}$$

for some $\lambda_0 > 1$. Accordingly, *if the conditions (6.6) and (6.9) hold then (6.4) implies (6.6.) as well as the dominated variation of* $1-F$ *and* $1-G$.

Our results permit various paraphrases of the sufficient conditions, and also of the ratio limit theorem itself. That (6.6) by itself is not sufficient is shown by (6.3*b*); without (6.9) certain subsequences may exhibit the pattern of slow variation, and the conclusion (6.5) must be replaced by a weaker conclusion of the form (6.3*b*).

7. On the tails of infinitely divisible distributions

To illustrate the usefulness of the notion of dominated variation in probabilistic contexts we prove the following

PROPOSITION. *Let H stand for an infinitely divisible probability distribution with Lévy measure* $M\{dx\}$. *If M varies dominatedly at* $+\infty$ *then*

$$1 - H(x) \sim M\{(x, \infty)\}, \qquad x \to +\infty \tag{7.1}$$

in the sense that the ratio of the two sides tends to unity at all points of continuity. (A very special case involving regular variation is mentioned in [1], p. 540.)

PROOF. We shall show that the general proposition follows easily from the special case where M is supported by the positive half axis and has a finite mass μ. In this case

$$M\{(x, \infty)\} = \mu[1-F(x)] \qquad x > 0 \tag{7.2}$$

where F is a probability distribution on $(0, x)$, and H reduces to the compound Poisson distribution given by

$$H(x) = e^{-\mu} \sum_{n=0}^{\infty} \frac{\mu^n}{n!} F^{n*}(x) \qquad x > 0\,. \tag{7.3}$$

We proceed to prove the assertion (7.1) for distributions of this form assuming that $1-F$ varies dominatedly. Note that F^{n*} is the distribution of the sum $S_n = X_1 + \ldots + X_n$ of n mutually independent random variables with the common distribution F. Since these variables are positive, the event $\{S_n > x\}$ occurs whenever at least one among the n variables exceeds x, and so

$$1 - F^{n*}(x) \geqslant n[1-F(x)] - \binom{n}{2}[1-F(x)]^2 \tag{7.4}$$