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ONE-SIDED ANALOGUES OF KARAMATA's REGULAR VARIATION *)

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To the memory of J. Karamata

1. Introduction

The monotone functions studied in this paper are assumed to be defined on $(0, \infty)$, and to be non-negative and right continuous. Point functions are introduced for notational convenience only, but we are really concerned with the associated measure which attributes value $U(x)$ to the interval $[0, x]$ when U increases, and to (x, ∞) when U decreases to zero.

Karamata's original theory of regularly varying functions has been generalized in chapter VIII of [1] to measures. A monotone function U *is said to vary regularly at infinity with exponent* α *if*

$$\lim_{t \to \infty} \frac{U(tx)}{U(t)} = x^{\alpha} \tag{1.1}$$

for some α and each $x > 0$. At first glance this definition appears to be artificial, but it is motivated by the fact that if the limit on the left exists, it is necessarily of the form x^{α}. Accordingly *U varies regularly at infinity if, and only if, as $t \to \infty$ the measures $U(t\mathrm{d}x)/U(t)$ converge to a finite measure* in every finite interval. (Here, and in the following, convergence of measures is taken in the usual weak sense.) The function U varies regularly at the origin if $U(t^{-1})$ varies regularly at infinity. From now on we omit the qualification " at infinity ", and it will be tacitly understood that in our passages to the limit the variable tends to ∞.

With an arbitrary measure U on $(0, \infty)$ we may associate the truncated moment functions

$$U_p(x) = \int_x^{\infty} y^{-p}\, U(dy) \tag{1.2}$$

* Work connected with a Project for research in probability theory at Princeton University, supported by the Army Research Office.

and

$$Z_q(x) = \int_0^x y^q\, U(dy) \tag{1.3}$$

defined whenever the integrals converge. These functions define new measures and they satisfy obvious identities such as

$$U_p(x) = \int_x^\infty y^{-p-q} Z_q(dy)\,. \tag{1.4}$$

For convenience of notation and exposition we shall from now on use $U = Z_0$ as representative of the whole family $\{Z_q\}$ and formulate all theorems in terms of U and U_p. Various relations between the diverse Z_q will be implicit in our theorems. As a last piece of notation we introduce the frequently occurring function

$$R_U(t) = \frac{t^p\, U_p(t)}{U(t)}\,. \tag{1.5}$$

The notion of regular variation was introduced, and achieved its greatest success, in connection with Tauberian theorems. In recent years more attention was paid to hitherto little known relations derived by Karamata in [3] and connecting the asymptotic behavior of the various functions Z_q and U_p. The basic theorems may be summarized as follows.

(i) *Let U vary regularly with exponent $\alpha \geqslant 0$. Then U_p exists for $p > \alpha$ (but for no $p < \alpha$). Furthermore*

$$\lim_{t \to \infty} R_U(t) = r \tag{1.6}$$

exists, and $r = \dfrac{\alpha}{p-\alpha} < \infty$. *If $r > 0$ then U_p varies regularly with exponent $\alpha - p$.*

(ii) *Let U be such that U_p exists for some fixed $p > 0$ and varies regularly with some exponent $q \leqslant 0$. Then the limit (1.6) exists, $0 < r \leqslant \infty$. If $r < \infty$ then U varies regularly with exponent $p + q$.*

(iii) *Let the limit r exist. If $0 < r < \infty$ then both U and U_p vary regularly with exponents*

$$\alpha = p\,\frac{r}{r+1} \quad \textit{and} \quad \alpha - p = -p\,\frac{1}{r+1} \tag{1.7}$$

respectively ; hence

$$(1.8) \qquad U_p(t) \sim rt^{-p}\, U(t)\,.$$

If $r=0$ *then* U *varies slowly* ($\alpha=0$) *and* $U_p(t)=o(t^{-p}\,U(t))$. *If* $r=\infty$ *then* U_p *varies slowly and* $U(\mathrm{t})=o(t^p\,U_p(t))$.

Thus except when either U or U_p varies slowly regular variation is tied to a relation of the form (1.8) and the existence of the limit (1.6) characterizes regular variation.

Karamata considered only measures defined by densities. Simplified proofs and extensions of his results can be found in [1]. In this book it was shown that the measure theoretic version of Karamata's relations introduces coherence and unity in the theory of domains of attraction, and that it leads to a substantial simplification of this theory. (In such connections U is usually the truncated second moment of a probability distribution, and U_2 is then the tail sum of this distribution.)

In [2] it turned out that various compactness arguments and local limit theorems in probability do not depend on the full strength of regular variation, but only on a one sided version of it.

We now proceed to describe this generalization and to show that Karamata's relations carry over to a surprising extent. In section 4 we discuss inequalities going in the opposite direction.

In section 5 we turn to *ratio limit theorems*. Roughly speaking, we show that if two monotone functions U and V stand in the relation $V=UL$ with L slowly varying, then also $V_p \sim U_p\,L$. For regularly varying function this is implicit in Karamata's relation (1.8), but it is surprising that dominated variation should suffice for the conclusion. Furthermore, we obtain a necessary and sufficient condition for the ratio V/U to be slowly varying. In section 6 these results are reformulated in the form of a *Tauberian ratio limit theorem*.

To illustrate the way in which dominated variation occurs naturally in probabilistic contexts we discuss in section 7 the asymptotic behavior of the *tails of infinitely divisible distributions*. This section is independent of the Karamata relations and may be read directly after section 2.

2. Dominated variation

We start from the following

Definition. *A monotone function U varies dominatedly if*

$$\text{(2.1)} \qquad \limsup \frac{U(2t)}{U(t)} < \infty \quad \textit{in case} \quad U \uparrow$$

$$\limsup \frac{U(t/2)}{U(t)} < \infty \quad \textit{in case} \quad U \downarrow .$$

This leads immediately to the

Criterion. *A non-decreasing U varies dominatedly if there exist constants* C, γ, *and* t_0 *such that*

$$\text{(2.2)} \qquad \frac{U(tx)}{U(t)} < Cx^{\gamma} \qquad x > 1 , \; t > t_0 .$$

For non-increasing U the same criterion applies with $x>1$ *replaced by* $x<1$.

Proof. The sufficiency is obvious. Assume (2.1) and choose t_0 and C such that

$$\text{(2.3)} \qquad \frac{U(2t)}{U(t)} < C \qquad \text{for} \qquad t > t_0 .$$

Put $\gamma = \mathrm{Log}_2\, C$. For $x>1$ define n by $2^{n-1} < x \leqslant 2^n$. A repeated application of (2.3) then shows that the left side in (2.2) is $\leqslant C^n \leqslant Cx^{\gamma}$.

Dominated variation of U may be described by saying that *the measures associated with* $U(t\cdot)/U(t)$ *form a sequentially compact family* in the sense that every sequence contains a subsequence converging on finite intervals to a finite measure. As in the case of slowly varying functions, the occurrence of limit measures that vanish identically on $(0, \infty)$ introduces some lack of symmetry. The supplementary condition (4.1) is designed to avoid this anomaly.

3. One-sided version of the Karamata relations

From now on U will stand for a non-decreasing function and $p>0$ will be a fixed number such that the integral U_p converges. Only the case $U(\infty) = \infty$ is of practical interest. We adhere to the notation (1.5) for R_U and put

$$\underline{r} = \liminf R_U(t), \qquad \bar{r} = \limsup R_U(t). \tag{3.1}$$

We shall also use the notation

$$\mathscr{I}_U(t) = \int_t^\infty y^{-p-1} U(y)\, dy. \tag{3.2}$$

Theorem 1. *For U to vary dominatedly it is necessary and sufficient that $\bar{r}<\infty$. Similarly U_p varies dominatedly iff $\underline{r}>0$.*

More precisely: The relation (2.2) with $\gamma<p$ entails

$$R_U(t) \leqslant A \qquad t > t_0 \tag{3.3}$$

with

$$A = \frac{Cp}{p-\gamma} - 1. \tag{3.4}$$

Conversely, (3.3) implies (2.2) with

$$C = A + 1, \qquad \gamma = \frac{A}{A+1} p. \tag{3.5}$$

In like manner, if

$$R_U(t) \geqslant \eta > 0, \qquad t > t_0 \tag{3.6}$$

then

$$\frac{U_p(tx)}{U_p(t)} \geqslant K x^{-q}, \qquad x > 1,\ t > t_0 \tag{3.7}$$

with

$$K = \frac{\eta}{\eta+1}, \qquad q = \frac{p}{\eta+1}. \tag{3.8}$$

Conversely, if (3.7) holds with $q<p$ then

$$r \geq \frac{K(p-q)}{Kq+(1-K)p}. \tag{3.9}$$

(Note that necessarily $K \leq 1$ as can be seen letting $x \to 1$ in (3.7). On replacing t by tx^{-1} it is seen that (3.7) not only asserts dominated variation of U_p, but implies uniformity away from the origin.)

PROOF. (i) Using integration by parts and the notation (3.2) it is seen that the definition (1.2) of U_p leads to the identity

$$p\,\mathscr{I}_U(t) = U_p(t) + t^{-p}\,U(t) \tag{3.10}$$

valid at all points of continuity. If (2.2) holds with $\gamma<p$ we conclude for $t>t_0$

$$U_p(t) + t^{-p}\,U(t) \leq Cp \cdot U(t) \int_t^{\infty} y^{-p-1}(y/t)^{\gamma}\,dy = \tag{3.11}$$

$$= C\frac{p}{p-\gamma}\,t^{-p}\,U(t)$$

and so (3.3) holds with A defined in (3.4).

(ii) Assume (3.3). Then by (3.10)

$$pt^p\,\mathscr{I}_U(t) \leq (A+1)\,U(t) \tag{3.12}$$

or

$$\frac{s^{-p-1}\,U(s)}{\mathscr{I}_U(s)} \geq \frac{p}{A+1}\cdot\frac{1}{s} \qquad s>t_0\,. \tag{3.13}$$

Integrating between t and $tx>t$ we get

$$\log\frac{\mathscr{I}_U(t)}{\mathscr{I}_U(tx)} \geq \frac{p}{A+1}\log x\,, \qquad t>t_0\,. \tag{3.14}$$

Thus from (3.12)

$$(A+1)\,t^{-p}\,U(t) \geq p\mathscr{I}_U(t) \geq p\mathscr{I}_U(tx)\cdot x^{p/(A+1)} \tag{3.15}$$

and by the definition (3.2)

$$p\mathscr{I}_U(tx) \geq U(tx)\cdot(tx)^{-p}\,. \tag{3.16}$$

Accordingly, (2.2) holds with C and γ given in (3.5). (This part of the theorem was proved slightly differently in [2].)

(iii) Assume (3.6). As in the last part we conclude

$$\log \frac{\mathscr{I}_U(t)}{\mathscr{I}_U(tx)} \leqslant \frac{p}{\eta+1} \log x\,, \qquad x > 1\,,\ t > t_0\,. \tag{3.17}$$

A repeated use of (3.10) now shows that

$$\begin{aligned} U_p(t) \leqslant p\mathscr{I}_U(t) \leqslant p\mathscr{I}_U(tx)\cdot x^{p/(\eta+1)} &= \\ = x^{p/(\eta+1)}\left[U_p(tx) + (tx)^{-p}\, U(tx)\right]. \end{aligned} \tag{3.18}$$

From (3.6) with t replaced by tx it is seen that the expression within brackets is $<(1+\eta^{-1})\, U_p(tx)$, and so the assertion concerning (3.7) is true.

(iv) Assume (3.7) with $q<p$. From the definition (1.2) of U_p we get by Fubini's theorem

$$p\int_0^t y^{p-1}\, U_p(y)\, dy \;=\; U(t) + t^p\, U_p(t) \tag{3.19}$$

which proves that the integral on the left converges for all $t>0$. Let B stand for the value of the left side when $t=t_0$. For $y>t_0$ we can apply (3.7) to conclude

$$\begin{aligned} U(t) + t^p\, U_p(t) \leqslant B + pK^{-1}\, U_p(t) \int_{t_0}^t y^{p-1}\,(t/y)^q\, dy &< \\ < B + \frac{p}{p-q}\, K^{-1}\, t^p\, U_p(t)\,. \end{aligned} \tag{3.20}$$

Divide this inequality by $U(t)$ and let $t\to\infty$. If $U(t)\to\infty$ we get the assertion (3.8). If $U(t)$ remains bounded there is nothing to be proved because (3.7) implies that $t^p\, U_p(t)$ increases at least as fast as t^{p-q}, and hence $\underline{r}=\infty$ whenever U is bounded.

NOTE. Our result lack the perfect symmetry of the original Karamata relations. Starting from (2.2) we get (3.3)-(3.4). However, when we apply the converse with these given values we get (2.2) in the weaker form with γ replaced by a constant $\gamma'>\gamma$. Examples given in [2] show that, in an obvious sense, this result is the best possible.

4. Other conditions

By theorem 1 both U and U_p vary dominatedly whenever $\underline{r}>0$ and $\bar{r}<\infty$. The next theorem gives even simpler criteria for dominated variation that remain applicable in the limiting situations $\underline{r}=0$ and $\bar{r}=\infty$.

Theorem 3. *If there exists a number $\xi>1$ such that*

$$\liminf \frac{U(\xi t)}{U(t)} > 1 \tag{4.1}$$

then U_p varies dominatedly. Similarly the relation

$$\liminf \frac{U_p(t/\xi)}{U_p(t)} > 1 \tag{4.2}$$

implies the dominated variation of U.

Proof. Clearly

$$\frac{t^p\, U_p(t)}{U(t)} \geqslant \frac{t^p\left[U_p(t)-U_p(t\xi)\right]}{U(t)} \geqslant \xi^{-p}\, \frac{U(t\xi)-U(t)}{U(t)}\,. \tag{4.3}$$

When the right side is bounded away from 0 this implies $\underline{r}>0$, and so U_p varies dominatedly by theorem 1.

We can go a step further. If, besides (4.1), it is known that U varies dominatedly with exponent $\gamma<p$, then $t^{-p}\,U(t)/U_p(t)$ is bounded away from 0, and hence the second inequality in (4.3) implies that

$$\liminf \frac{U_p(t)-U_p(t\xi)}{U_p(t)} > 0\,. \tag{4.4}$$

This is equivalent to (4.2).

Similarly

$$\frac{U_p(t)-U_p(t\xi)}{U_p(t\xi)} \leqslant t^{-p}\, \frac{U(t\xi)-U(t)}{U_p(t\xi)} = \tag{4.5}$$

$$= \frac{\xi^p}{R_U(t\xi)}\, \frac{U(t\xi)-U(t)}{U(t\xi)}\,.$$

The second fraction on the right does not exceed 1, and so (4.2) ensures that $R_U(t\xi)$ remains bounded, and hence that U is of dominated variation.

Again, if it is known that R_U is bounded away from 0 then (4.5) shows that (4.2) implies (4.1).

We have thus proved the

COROLLARY. *If U is of dominated variation with exponent $\gamma<p$ then (4.1) implies (4.2). Similarly, if U_p is of dominated variation with exponent $-q$ where $q<p$, then (4.2) entails (4.1).* (In each case both functions are of dominated variation.)

5. RATIO LIMIT THEOREMS

Let U and V be non-decreasing unbounded functions, and suppose that L is slowly varying (= regularly varying with exponent 0).

DEFINITION. *We shall say that U and V are L-equivalent and write*

$$V \leftrightarrow UL \tag{5.1}$$

if the ratio UL/V tends to 1 at all points of continuity.

More precisely, it is required that for each $\varepsilon>0$ and fixed $\lambda>1$

$$(1-\varepsilon)\, L(t)\, U(t/\lambda) \leqslant V(t) \leqslant (1+\varepsilon)\, L(t)\, U(t\lambda) \tag{5.2}$$

for all t sufficiently large.

THEOREM 4. *Let U be of dominated variation. In order that there exist a slowly varying function* L *such that (5.1) holds it is necessary and sufficient that*

$$R_U(t) - R_V(t) \to 0 \qquad \textit{boundedly}. \tag{5.3}$$

Needless to say, R_V and $\mathscr{I}_V$ are defined by analogy with R_U in (1.5) and $\mathscr{I}_U$ in (3.2).

PROOF. (a) *Necessity.* Assume (5.1) and suppose that U satisfies the basic inequality (2.2). Obviously the slow variation of L implies that for t sufficiently large and all $x>1$

$$\frac{V(tx)}{V(t)} < C' x^{\gamma'} \tag{5.4}$$

for any pair of constants $C'>C$ and $\gamma'>\gamma$. Thus V is of dominated variation, and since $p>\gamma$ the function V_p exists.

Let $t_n \to \infty$ in such a way that the measures associated with $U(t_n \cdot)/U(t_n)$ tend (in finite intervals) to a limit measure m. The relation (5.1) implies obviously that the measures associated with $V(t_n \cdot)/V(t_n)$ tend to the same limit m. Thus when t runs through $\{t_n\}$ we have for fixed $x > 1$

$$(5.5) \qquad \frac{U_p(t) - U_p(tx)}{U(t)\, t^{-p}} = \int_1^x y^{-p} \frac{U(t\,dy)}{U(t)} \to \int_1^x y^{-p}\, m(dy),$$

and the same relation holds with U replaced by V. But (5.4) implies that this passage to the limit is uniform as $x \to \infty$; it remains valid also for $x = \infty$ with the right side being finite. We have thus shown that $R_U(t_n) - R_V(t_n) \to 0$. But the t_n may be picked as elements of an arbitrarily prescribed sequence, and so the limit relation in (5.3) holds pointwise for an arbitrary approach $t \to \infty$. Now we know that the dominated variation of U and V implies the boundedness of both R_U and R_V, and the condition (5.3) holds true.

(b) *Sufficiency.* The variation of U being dominated, R_U remains bounded and so (5.3) implies the boundedness of R_V and hence the dominated variation of V. The calculation of part (ii) in section 3 show that

$$(5.6) \qquad \frac{s^{-p-1}\, U(s)}{\mathcal{I}_U(s)} - \frac{s^{-p-1}\, V(s)}{\mathcal{I}_V(s)} = \frac{p}{t}\left[\frac{1}{1 + R_U(s)} - \frac{1}{1 + R_V(s)}\right].$$

The expression within brackets is in absolute value bounded by $|R_U(s) - R_V(s)|$, and therefore tends to 0 boundedly. Integrating between t and $tx > t$ we conclude therefore that

$$(5.7) \qquad \log \frac{\mathcal{I}_U(t)}{\mathcal{I}_U(tx)} \cdot \frac{\mathcal{I}_V(tx)}{\mathcal{I}_U(t)} \to 0\,.$$

In other words, the ratio $\mathcal{I}_U/\mathcal{I}_V$ varies slowly, and therefore we can put

$$(5.8) \qquad \mathcal{I}_V(t) = L(t)\, \mathcal{I}_U(t)$$

where L varies slowly.

We now recall the inequality (3.14) which implies that to each $\lambda > 1$ there exists an $\eta < 1$ such that

$$(5.9) \qquad \mathcal{I}_U(\lambda t) < \eta\, \mathcal{I}_U(t)$$

for all t sufficiently large. From (5.8) we conclude therefore that

$$(5.10) \qquad \lim \frac{\mathscr{I}_V(t) - \mathscr{I}_V(\lambda t)}{[\mathscr{I}_U(t) - \mathscr{I}_U(\lambda t)]\, L(t)} =$$

$$= \lim \frac{L(t)\, \mathscr{I}_U(t) - L(\lambda t)\, \mathscr{I}_U(\lambda t)}{L(t)\, \mathscr{I}_U(t) - L(t)\, \mathscr{I}_U(\lambda t)} = 1 .$$

But the fraction on the left lies between

$$\frac{V(\lambda t)}{U(t)\, L(t)} \quad \text{and} \quad \frac{V(t)}{U(\lambda t)\, L(t)}$$

and so (5.1) is true.

6. Application to Tauberian theorems

If the measure U varies regularly at infinity, then its Laplace transform ω varies regularly at the origin. More precisely, Karamata's now classical Tauberian theorem states that for any $\alpha \geqslant 0$ and slowly varying function L the two relations

$$(6.1) \qquad U(x) \sim x^{\alpha} L(x) \qquad\qquad \omega(\lambda) \sim \Gamma(\alpha+1)\, \lambda^{-\alpha} L(\lambda^{-1})$$

imply each other; here $x \to \infty$ but $\lambda \to 0$. [The sign $\sim$ indicates that the ratio of the two sides tends to 1.] For an example of a probabilistic application suppose that

$$(6.2) \qquad U(x) = \int_0^x y^p F(dy)$$

is the truncated p^{th} moment of a probability distribution F on the positive half axis. For simplicity let p stand for a positive integer. Then $U_p(x) = 1 - F(x)$ and $\omega = (-1)^p \phi^{(p)}$ where ϕ is the Laplace-Stieltjes transform of F. If ω varies regularly in accordance with (6.1) then Karamata's relation (1.8) implies that

$$(6.3a) \qquad 1 - F(x) \sim \frac{\alpha}{p - \alpha}\, x^{\alpha - p} L(x) \qquad \textit{when} \qquad \alpha < p$$

$$(6.3b) \qquad 1 - F(x) = o(x^{\alpha} L(x)) \qquad \textit{when} \qquad \alpha = p .$$

(Note that necessarily $0 \leqslant \alpha \leqslant p$ because the measure F is finite.) In other words, the behavior at the origin of the derivatives of the Laplace transform determines the behavior of the tail $1 - F(x)$, and vice versa.

From the continuity theorem for Laplace transforms one concludes without difficulties that *dominated variation of U at* ∞ *is equivalent to dominated variation of* ω *at* 0. Even in the case of mere dominated variation the behavior of $\omega = (-1)^p \phi^{(p)}$ therefore permits inferences concerning the behavior of the tail $1-F$, but naturally the conclusions will lack the pleasing precision of (6.3). It is therefore remarkable that precise asymptotic equivalence relations can be obtained when comparing two probability distributions F and G with Laplace transforms ϕ and γ.

A typical *Tauberian ratio limit theorem would state that the two relations*

(6.4) $$\gamma^{(p)}(\lambda) \sim \phi^{(p)}(\lambda)\, L\left(\frac{1}{\lambda}\right) \qquad \lambda \to 0$$

and

(6.5) $$1 - G(x) \sim [1 - F(x)]\, L(x) \qquad x \to \infty$$

imply each other. This is not true in full generality; indeed, (6.3*b*) points to exceptional situations even when the transforms in (6.4) vary regularly. However, our results yield a variety of fairly general sufficient conditions for the validity of the conclusion. Suppose, for example, that for some constants A and $\alpha < p$

(6.6) $$(-1)^p \phi^{(p)}(\lambda) < A\,\lambda^{-\alpha}$$

for λ sufficiently small. It is easily seen in this case that U varies dominatedly with exponent α and (6.4) is equivalent to

(6.7) $$V \leftrightarrow UL$$

in the sense of (5.1). (Here V stands for the truncated moment function of G defined as in (6.2).) Theorem 4 then asserts that (5.3) holds, and this implies

(6.8) $$V_p \leftrightarrow U_p L$$

whenever U is bounded away from 0. Now (6.5) differs from (6.8) only notationally, and we know that the condition (4.1) guarantees that R_U is bounded away from 0 and that $U_p = 1 - F$ varies dominatedly. Again, (4.1) holds if, and only if, each limit of a convergent sequence of measures $U(t_n\,dx)/U(t_n)$ attributes a positive measure to $(0, \infty)$. This requirement is satisfied if, and only if,

(6.9) $$\liminf_{\varepsilon \to 0} \frac{\phi^{(p)}(\varepsilon\lambda_0)}{\phi^{(p)}(\varepsilon)} < 1$$

for some $\lambda_0 > 1$. Accordingly, *if the conditions (6.6) and (6.9) hold then (6.4) implies (6.6.) as well as the dominated variation of* $1-F$ *and* $1-G$.

Our results permit various paraphrases of the sufficient conditions, and also of the ratio limit theorem itself. That (6.6) by itself is not sufficient is shown by (6.3*b*); without (6.9) certain subsequences may exhibit the pattern of slow variation, and the conclusion (6.5) must be replaced by a weaker conclusion of the form (6.3*b*).

7. On the tails of infinitely divisible distributions

To illustrate the usefulness of the notion of dominated variation in probabilistic contexts we prove the following

Proposition. *Let H stand for an infinitely divisible probability distribution with Lévy measure M* $\{dx\}$. *If M varies dominatedly at* $+\infty$ *then*

(7.1) $$1 - H(x) \sim M\{(x, \infty)\}, \qquad x \to +\infty$$

in the sense that the ratio of the two sides tends to unity at all points of continuity. (A very special case involving regular variation is mentioned in [1], p. 540.)

Proof. We shall show that the general proposition follows easily from the special case where M is supported by the positive half axis and has a finite mass μ. In this case

(7.2) $$M\{(x, \infty)\} = \mu[1-F(x)] \qquad x > 0$$

where F is a probability distribution on $(0, x)$, and H reduces to the compound Poisson distribution given by

(7.3) $$H(x) = e^{-\mu} \sum_{n=0}^{\infty} \frac{\mu^n}{n!} F^{n*}(x) \qquad x > 0 .$$

We proceed to prove the assertion (7.1) for distributions of this form assuming that $1-F$ varies dominatedly. Note that F^{n*} is the distribution of the sum $S_n = X_1 + \ldots + X_n$ of n mutually independent random variables with the common distribution F. Since these variables are positive, the event $\{S_n > x\}$ occurs whenever at least one among the n variables exceeds x, and so

(7.4) $$1 - F^{n*}(x) \geqslant n[1-F(x)] - \binom{n}{2}[1-F(x)]^2$$

by an easily verified inequality named after Bonferoni. Substituting into (7.3) it follows that

$$(7.5) \qquad 1 - H(x) \geqslant \mu[1-F(x)] - \frac{1}{2}\mu^2[1-F(x)]^2 = \\ = M\{(x, \infty)\}[1+o(1)], \qquad x \to \infty .$$

To obtain an appraisal in the opposite direction choose $0<\varepsilon<\frac{1}{2}$ and note that the event $\{S_n>x\}$ cannot occur unless either at least one among the variables $X_1, \ldots, X_n$ exceeds $(1-\varepsilon)x$, or at least two among them exceed $\varepsilon x/n$. Thus

$$(7.6) \qquad 1 - F^{n*}(x) \leqslant n[1-F((1-\varepsilon)x)] + \binom{n}{2}[1-F(\varepsilon x/n)]^2 .$$

To apply the argument used in (7.5) we would have to know that the ratio of the two brackets on the right tends to 0 as $x\to\infty$. Because of the assumed dominated variation of $1-F$ this is true for every fixed n, but to make the ratio $<\delta$ we must have $\varepsilon x/n$ sufficiently large, that is, $n\leqslant ax$, where a is an appropriate constant. On the other hand, if r is the smallest integer exceeding ax and if $ax>2\mu$ we have trivially

$$(7.7) \qquad \sum_{n=r}^{\infty}\frac{\mu^n}{n!} < 2\left(\frac{\mu}{r}\right)^r < 2\left(\frac{\mu}{ax}\right)^{ax}$$

and the right side tends to zero faster than any power of x^{-1}. In view of the dominated variation of $1-F$ this implies that the quantity (7.7) is $o(1-F(x))$, and this together with (7.6) shows as in (7.5) that

$$(7.8) \qquad 1 - H(x) \leqslant \mu[1-F(x)](1+o(1)) .$$

This proves the assertion for distributions of the form (7.3).

For the general case we represent the Lévy measure M as a sum of three measures supported by the intervals $(1, \infty)$, $[-1, 1)$, and $(-\infty, 1]$, respectively. This puts H in the form of a triple convolution, and so we may conceive of H as of the distribution of a sum $X+Y+Z+$const. of three infinitely divisible mutually independent random variables such that $X\geqslant 0$, $Z\leqslant 0$, and Y has a Lévy measure supported by $[-1, 1]$. It follows that Y has moments of all orders, and hence for arbitrary $\varepsilon>0$ and n

$$(7.9) \qquad P\{|Y| > \varepsilon x\} = o(x^n) \qquad x \to \infty .$$

Because of the assumed dominated variation $M\{(x, \infty)\}$ decreases more slowly than a certain power $x^{-\alpha}$, and hence the quantity (7.9) is $o(\mathrm{M}\{(ax, \infty)\}$ for any fixed $a>0$. Since $Z \leqslant 0$ and X has a distribution of the form (7.3) we conclude that

$$(7.10) \qquad P\{X + Y + Z > x\} \leqslant P\{X > (1-\varepsilon)x\} + P\{Y > \varepsilon x\} \sim \sim M\{((1-\varepsilon)x, \infty)\}.$$

On the other hand,

$$(7.11) \qquad P\{X + Y + Z > x\} \geqslant P\{X > (1+\varepsilon)x\} \cdot P\{Y + Z > -\varepsilon x\},$$

and the last factor tends to 1 as $x \to \infty$. The probabilities on the left are therefore $\sim M\{(x, \infty)\}$, as asserted.

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