§6. The Fourier transform of functions on R^n

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **14 (1968)**

Heft 1: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **26.09.2024**

where

$$t = \frac{\chi_2 \bar{x}_2}{|x|^2}.$$ [1]

§ 6. The Fourier transform of functions on $\mathbf{R}^n$

We have shown that $L^2(\Sigma_{n-1})$ can be decomposed into a direct sum of mutually orthogonal subspaces (the spaces $\mathscr{H}_n^{(k)}$) that are invariant and irreducible under the action of rotations. There exists a corresponding decomposition of $L^2(\mathbf{R}^n)$ and the spaces making up this decomposition are intimately connected with the Fourier transform of functions of n real variables. In this section we shall construct these spaces and study the action of the Fourier transform restricted to them. We shall see that also in this situation the rotation group $SO(n)$ and its representations play a central role.

If f belongs to $L^1(\mathbf{R}^n)$ its *Fourier transform* $\hat{f}$ is defined by letting

$$(\mathscr{F} f)(y) = \hat{f}(y) = \int_{\mathbf{R}^n} f(x)\, e^{-2\pi i x.y}\, dx$$

for $y \in \mathbf{R}^n$.[1])

Perhaps the simplest class of functions that is invariant under the action of the Fourier transform is the collection of *radial functions.* We recall that these are the functions on $\mathbf{R}^n$ that depend only on $|x|$; equivalently, f is radial if $\rho_v f = f$ for all $v \in SO(n)$, where the operator ρ_v is defined by

$$(\rho_v f)(x) = f(v^{-1} x)$$

for all $x \in \mathbf{R}^n$. Since Lebesgue measure is invariant under the action of rotations and $v = v^*$ when $v \in SO(n)$,

$$\int_{\mathbf{R}^n} f(x)\, e^{-2\pi i x.v^{-1}y}\, dx = \int_{\mathbf{R}^n} f(x)\, e^{-2\pi i v x.y}\, dx = \int_{\mathbf{R}^n} f(v^{-1}x)\, e^{-2\pi i x.y}\, dx.$$

That is,

$$(\mathscr{F} \rho_v) f = (\rho_v \mathscr{F}) f \tag{6.1}$$

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1) It is not hard to use these results in order to obtain analogous results for $SO(3)$. We refer the reader to Vilenkin [11] for complete details.

1) When $f \in L^2(\mathbf{R}^n)$ the integral defining $\hat{f}$ is not defined in the Lebesgue sense. In this case, $\hat{f}$ is usually defined as the limit in the L^2 mean of the sequence $\hat{f}^k(y) = \int_{|x| \leq k} f(x)\, e^{-2\pi i x \cdot y}\, dx$. In order to avoid technical difficulties that arise from this definition we shall restrict our attention to integrable functions

for all $f \in L^1(\mathbf{R}^n)$. This basic property, that Fourier transformation commutes with the action of rotations clearly implies.

THEOREM (6.2). *If* $f \in L^1(\mathbf{R}^n)$ *is radial then* $\hat{f}$ *is also a radial function.*

In order to extend this invariance property we introduce, for $k = 0, 1, 2, \ldots$, the class of functions $\mathfrak{h}^{(k)} = \mathfrak{h}_n^{(k)}$ mapping $\mathbf{R}^n$ into $\mathbf{C}^{d_k}$ having the form

$$F(x) = f(|x|)\left(Y_1(\xi), \ldots, Y_{d_k}(\xi)\right) = \left(F_1(x), \ldots, F_{d_k}(x)\right),$$

where

$$x = |x|\,\xi\,, \int_0^\infty f(r)\, r^{n-1}\, dr < \infty \;^{1)} \; \textit{and} \; \{Y_1, Y_2, \ldots, Y_{d_k}\}$$

is an orthonormal basis of $\mathscr{H}_n^{(k)}$ such that $Y_1 = a_k^{-1} Z_1$ (that is, orthonormality is to be taken with respect to the inner product (2.6)). Such a basis was considered, for example, in theorem (2.16). When $k = 0$ this class is precisely the set of radial functions. It will be convenient if we choose the $Y_1, \ldots Y_{d_k}$ to be real-valued.

Let $T^{(k)} = (t_{lj}^{(k)})$ be the matrix of the representation $S^{k,n}$ with respect to the basis $\{Y_1, Y_2, \ldots, Y_{d_k}\}$; that is, the functions $t_{lj}^{(k)} = t_{lj}$ satisfy

$$(S_v^{k,n} Y_j)(\xi) = Y_j(v^{-1}\xi) = \sum_{l=1}^{d_k} t_{lj}(v)\, Y_l(\xi)$$

for $j = 1, 2, \ldots, d_k$. If we let

$$\rho_v F = (\rho_v F_1, \ldots, \rho_v F_{d_k})$$

we then have

$$(\rho_v F)(x) = f(|x|)\left(Y_1(v^{-1}\xi), \ldots, Y_{d_k}(v^{-1}\xi)\right) =$$

$$= f(|x|)\left(Y_1(\xi), \ldots, Y_{d_k}(\xi)\right) \begin{pmatrix} t_{11}(v) & \ldots & t_{1d_k}(v) \\ t_{21}(v) & \ldots & t_{2d_k}(v) \\ \ldots & \ldots & \ldots \\ t_{d_k 1}(v) & \ldots & t_{d_k d_k}(v) \end{pmatrix} = (T_v^{(k)} F)(x)\,,$$

The last equality being the definition of the operator $T_v^{(k)}$ acting on F. That is,

$$\rho_v F = T_v^{(k)} F \tag{6.3}$$

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1) This condition merely assures us that the radial function $g(x) = f(|x|)$ is integrable on $\mathbf{R}^n$.

for all $v \in SO(n)$. If we now apply the Fourier transform to each component of $\rho_v F$, it follows from (6.2) and (6.3) that

(6.4) $$\rho_v \hat{F} = \rho_v(\hat{F}_1, ..., \hat{F}_{d_k}) = T_v^{(k)} \hat{F} .$$

The following, together with relation (6.4), shows that $\hat{F}$ must have the same form as F; that is,

(6.5) $$\hat{F}(y) = \tilde{f}(|y|)(Y_1(\eta), ..., Y_{d_k}(\eta))$$

for all $y = |y| \eta \in \mathbf{R}^n$.

THEOREM (6.6). *Suppose* $G = (G_1, ..., G_{d_k})$ *is a continuous function mapping* $\mathbf{R}^n$ *into* $\mathbf{C}^{dk}$ *such that*

(6.7) $$\rho_v G = T_v^{(k)} G$$

for all $v \in SO(n)$, *then*

$$G(y) = a_k^{-1} G_1(|y|\mathbf{1})(Y_1(\eta), ..., Y_{d_k}(\eta))$$

for all $y = |y|$ *in* $\mathbf{R}^n$.

Proof. Let $v \in SO(n)$ be such that $y = |y| v' \mathbf{1} = |y| v^{-1} \mathbf{1}$. Then, by (6.7)

$$G(y) = G(v^{-1}|y|\mathbf{1}) = (T_v^{(k)} G)(|y|\mathbf{1}) .$$

Consequently,

(6.8) $$G_j(y) = \sum_{l=1}^{d_k} t_{lj}(v) G_l(|y|\mathbf{1})$$

for $j = 1, 2, ..., d_k$. If $u \in SO(n-1)$ then $y = |y| v^{-1}\mathbf{1} = y = |y| v^{-1} u^{-1} \mathbf{1} = |y| (uv)^{-1} \mathbf{1}$; thus, if we replace v by uv in (6.8) we obtain

$$G_j(y) = \sum_{l=1}^{d_k} t_{lj}(uv) G_l(|y|\mathbf{1}) .$$

Integrating over $SO(n-1)$, therefore,

$$G_j(y) = \sum_{l=1}^{d_k} G_l(|y|\mathbf{1}) \int_{SO(n-1)} t_{lj}(uv)\, du .$$

But, by (3.2) and theorem (3.5) (or (3.15))

$$\int_{SO(n-1)} t_{lj}(uv)\,du = \begin{cases} t_{lj}(v) & \textit{when} \quad l = 1 \\ 0 & \textit{when} \quad l > 1 \end{cases}.$$

This equality and (2.17) show that

$$G_j(y) = G_j(v^{-1}|y|\mathbf{1}) = G_1(|y|\mathbf{1})\,\overline{Y_j(v^{-1}\mathbf{1})}\,a_k^{-1}.$$

(Since

$$\overline{t_{lj}(v)} = t_{jl}(v^{-1}) = \overline{Y_j(v^{-1}\mathbf{1})}\,).$$

Writing $y = |y|\,\eta$, where $\eta = v^{-1}\,\mathbf{1}$, and using the fact that Y_j is real-valued, we obtain the desired result

$$G_j(y) = a_k^{-1}\,G_1(|y|\mathbf{1})\,Y_j(\eta),$$

$j = 1, 2, \ldots, d_k$.

THEOREM (6.9). *Let* Y *be a spherical harmonic of degree* k *and* f *a function on* $(-\infty, \infty)$ *satisfying*

(*i*) $$\int_0^\infty |f(r)|\,r^{n-1}\,dr < \infty.$$

If $h(x) = f|x|)\,Y(\xi)$, *when* $x = |x|\,\xi \in \mathbf{R}^n$, *then* $h \in L^1(\mathbf{R}^n)$ *and*

$$\hat{h}(y) = \tilde{f}\,(|y|\;Y(\eta)$$

for all $y = |y|\,\eta \in \mathbf{R}^n$. *The transformation* $f \to \tilde{f}$ *depends only on* k *and* n *and, in particular, is independent of* $Y \in \mathscr{H}_n^{(k)}$.

Proof. Let $\{Y_1, \ldots, Y_{d_k}\}$ be the basis of $\mathscr{H}_n^{(k)}$ that was used in the previous theorem and $F(x) = (f(|x|)\,Y_1(\xi), \ldots, f(|x|)\,Y_{d_k}(\xi)) = (F_1(x), \ldots, F_{d_k}(x))$. Condition (i) guarantees that each of the functions $F_j, j = 1, \ldots, d_k$, is integrable.[1]) Thus, $\hat{F} = (\hat{F}_1, \ldots, \hat{F}_{d_k})$ is well defined, continuous (as can be very easily shown), and satisfies relation (6.4). By theorem (6.6), therefore,

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1) Using polar coordinates $x = |x|\,\xi$, with $\xi \in \Sigma_{n-1}$, we have $\int_{\mathbf{R}^n} |F_j(x)|\,dx = \int_{\Sigma_{n-1}} \omega_{n-1} \{\int_0^\infty |f(r)|\,r^{n-1}\,dr\}\,|Y_j(\xi)|\,d\xi < \infty$, where ω_{n-1} is the "area" of Σ_{n-1}.

$$\hat{F}(y) = a_k^{-1} \hat{F}_1(|y|\mathbf{1})(Y_1(\eta), \ldots, Y_{d_k}(\eta)) \quad \textit{for} \quad y = |y|\eta \in \mathbf{R}^n .$$

Putting $\tilde{f}(|y|) = a_k^{-1} \hat{F}_1(|y|\mathbf{1})$ we obtain equality (6.5). Since $\{Y_1, \ldots, Y_{d_k}\}$ is a basis of $\mathscr{H}_n^{(k)}$ we can find coefficients $b_1, \ldots, b_{d_k}$ such that

$$Y = \sum_{l=1}^{d_k} b_l Y_l .$$

Thus,

$$h(x) = \sum_{l=1}^{d_k} f(|x|) b_l Y_l(\xi) .$$

We have just shown that the Fourier transform of $F_l(x) = f(|x|) Y_l(\xi)$ has the values $\tilde{f}(|y|) Y_l(\eta)$. Thus,

$$\hat{h}(y) = \sum_{l=1}^{d_k} b_l \tilde{f}(|y|) Y_l(\eta) = \tilde{f}(|y|) \sum_{l=1}^{d_k} b_l Y_l(\eta) = \tilde{f}(|y|) Y(\eta) .$$

This proves the theorem.

It is not hard to give an explicit form for the mapping $f \to \tilde{f}$ in terms of the *Bessel functions*

$$J_\lambda(t) = \frac{(t/2)^\lambda}{\Gamma\left(\frac{2\lambda+1}{2}\right)\Gamma\left(\frac{1}{2}\right)} \int_{-1}^{1} e^{its}(1-s^2)^{\frac{2\lambda-1}{2}} ds .$$

We shall show, in fact, that

$$\tilde{f}(t) = \gamma_{k,n} t^{\frac{2-n}{2}} \int_0^\infty f(r) J_{k+\frac{n-2}{2}}(2\pi t r) r^{n/2} dr . \quad (6.10)$$ [1]

Since $\tilde{f}$ is independent of $Y \in \mathscr{H}_n^{(k)}$ let us choose $h(x) = f(|x|) Z_1(\xi) = f(|x|) P^{(k)}(\xi.\mathbf{1})$. Then

$$\hat{f}(y) = \int_{\mathbf{R}^n} e^{-2\pi i y.x} f(|x|) P^{(k)}(\xi\cdot\mathbf{1}) dx =$$

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1) We shall not calculate $\gamma_{k,n}$. The fact that this constant equals $2\pi i^{-k}$ can be shown by evaluating the integral in (6.10) when $f(r) = e^{-r^2}$ (see Stein and Weiss [10], Chapter IV, section 3) or by using the constants obtained below.

$$= \omega_{n-1} \int_0^\infty r^{n-1} f(r) \{ \int_{\Sigma_{n-1}} e^{-2\pi i |y| r(\eta \cdot \xi)} P^{(k)}(\xi.\mathbf{1})\, d\xi \}\, dr$$

Writing $y = t\,\eta$, this means that we have to compute

$$\int_{\Sigma_{n-1}} e^{-2\pi i r t(\eta\cdot\xi)} P^{(k)}(\xi.\mathbf{1})\, d\xi\,.$$

But, by the Funk-Hecke theorem (4.16) this integral is equal to

$$P^{(k)}(\eta.\mathbf{1})\, a_k^{-2}\, c_n \int_{-1}^{1} e^{-2\pi i r t s} P^{(k)}(s)(1-s^2)^{\frac{n-3}{2}}\, ds\,.$$

On the other hand, by (4.4), and, then integrating by parts k times we have

$$\int_{-1}^{1} e^{-2\pi i r t s} P^{(k)}(s)(1-s^2)^{\frac{n-3}{2}}\, ds = \alpha_{k,n} \int_{-}^{1} e^{-2\pi i r t s} \left[\frac{d^k}{dt^k}(1-s^2)^{k+\frac{n-3}{2}}\right] ds$$

$$= \beta_{k,n} \int_{-} (rt)^k e^{2\pi i r t s} (1-s^2)^{k+\frac{n-3}{2}}\, ds\,.$$

The last integral, however, is the one involved in the definition of J_λ when $\lambda = (2k+n-2)/2$. Equality (6.10) now follows immediately.[1])

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1) The Bessel functions we have encountered here arise in much the same way as did the ultraspherical Polynomials. Instead of the group *SO* (*n*), however, one must study the group of all rigid motions on $\mathbf{R}^{(n)}$ (see VILENKIN [11] for details).