

Direct images of sheaves

Objektyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **14 (1968)**

Heft 1: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **19.09.2024**

Nutzungsbedingungen

Die ETH-Bibliothek ist Anbieterin der digitalisierten Zeitschriften. Sie besitzt keine Urheberrechte an den Inhalten der Zeitschriften. Die Rechte liegen in der Regel bei den Herausgebern.

Die auf der Plattform e-periodica veröffentlichten Dokumente stehen für nicht-kommerzielle Zwecke in Lehre und Forschung sowie für die private Nutzung frei zur Verfügung. Einzelne Dateien oder Ausdrucke aus diesem Angebot können zusammen mit diesen Nutzungsbedingungen und den korrekten Herkunftsbezeichnungen weitergegeben werden.

Das Veröffentlichen von Bildern in Print- und Online-Publikationen ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Die systematische Speicherung von Teilen des elektronischen Angebots auf anderen Servern bedarf ebenfalls des schriftlichen Einverständnisses der Rechteinhaber.

Haftungsausschluss

Alle Angaben erfolgen ohne Gewähr für Vollständigkeit oder Richtigkeit. Es wird keine Haftung übernommen für Schäden durch die Verwendung von Informationen aus diesem Online-Angebot oder durch das Fehlen von Informationen. Dies gilt auch für Inhalte Dritter, die über dieses Angebot zugänglich sind.

spread holomorphically, i.e. for any $x \in X$ there exists $f_1 \dots f_N \in I(X)$ such that x is an isolated common zero of $f_1 \dots f_N$.

Let X be a complex analytic manifold. A Stein covering $\mathfrak{U} = \{U_i\}_{i \in J}$ of X is an open covering of X such that every U_i is Stein. We shall often use the following result:

Leray's Theorem: If \mathfrak{U} is a Stein covering of X then $H^l(\mathfrak{U}, S) \rightarrow H^l(X, S)$ is an isomorphism for every coherent analytic sheaf S .

The isomorphism between $H^l(\mathfrak{U}, S)$ and $H^l(X, S)$ means the following: If $\underline{\xi} \in H^l(X, S)$ there exists $\xi \in Z^l(\mathfrak{U}, S)$ such that ξ maps into $\underline{\xi}$ under the natural homomorphism $Z^l(\mathfrak{U}, S) \rightarrow H^l(X, S)$ and moreover if $\underline{\xi} \in Z^l(\mathfrak{U}, S)$ is mapped into zero in $H^l(X, S)$ there exist $\eta \in C^{l-1}(\mathfrak{U}, S)$ such that $\xi = \delta\eta$ in $C^l(\mathfrak{U}, S)$.

DIRECT IMAGES OF SHEAVES

Let X and Y be complex analytic manifolds. Let $\psi : X \rightarrow Y$ be a holomorphic map and let S be an analytic sheaf on X . Now X is fibered by the fibers $X(y) = \psi^{-1}(y)$ for $y \in Y$. Let U be an open neighborhood of a point $y \in Y$, then $V = \psi^{-1}(U)$ is an open set in X . Hence V is a complex analytic manifold and the restriction of S to V gives an analytic sheaf on V . We can now define $H^l(V, S)$. Let us put $H_y^l = \bigcup_U H^l(\psi^{-1}(U), S)$

where U runs over all open neighborhoods of y in Y . In H_y^l we introduce an equivalence relation as follows: $\xi_1 \in H^l(\psi^{-1}(U_1), S)$ and $\xi_2 \in H^l(\psi^{-1}(U_2), S)$ are equivalent iff there exists $U = U(y)$ in Y such that $U \subset U_1 \cap U_2$ and $\xi_1|_{\psi^{-1}(U)} = \xi_2|_{\psi^{-1}(U)}$ in $H^l(\psi^{-1}(U), S)$. We let $\psi_{(l)}(S)_{(y)}$ denote the set of equivalence classes in H_y^l . The equivalence class generated by $\xi \in H^l(\psi^{-1}(U), S)$ is denoted by ξ_y . The set $\psi_{(l)}(S)_{(y)}$ is called the set of germs of cohomology classes of dimension l along the fiber $X(y)$. Now $\psi_{(l)}(S)_{(y)}$ is an $\mathcal{O}_{y,Y}$ -module. For if $g_y \in \mathcal{O}_{y,Y}$ we have a representative $g \in I(U)$ for some open neighborhood U of y . Then $g \circ \psi \in I(\psi^{-1}(U))$. If $\xi_y \in \psi_{(l)}(S)_{(y)}$ and U is sufficiently small we can find a representative $\xi \in H^l(\psi^{-1}(U), S)$ for ξ_y . Then we put $g_y \cdot \xi_y = ((g \circ \psi)\xi)_y$. Now we form $\psi_{(l)}(S) = \bigcup_{y \in Y} \psi_{(l)}(S)_{(y)}$ where we introduce a sheaf topology.

A base of the open sets are $\{\xi_y : y \in U\}$ for $\xi \in H^l(\psi^{-1}(U), S)$. If $\xi \in H^l(X, S)$ then the map $y \rightarrow \xi_y$ is a cross-section in $\psi_{(l)}(S)$. We call it the direct image of ξ and denote it by $\psi_{(l)}(\xi)$. The sheaf $\psi_{(l)}(S)$ is the direct

image sheaf of S of dimension l . Our main problem is to decide whether $\psi_{(l)}(S)$ is a coherent analytic sheaf of \mathcal{O}_Y -modules if S is a coherent analytic sheaf on X .

A VERY SPECIAL CASE

We shall consider a special case where our main problem is easily solved. Let X_0 be a compact analytic manifold of pure dimension $m - n$. We put $E^n(\rho_0) = \{ (t_1 \dots t_n) \in \mathbf{C}^n ; |t_i| < \rho_i^0 \}$. Here $\rho_0 = (\rho_1^0 \dots \rho_n^0)$ is a fixed n -tuple of strictly positive numbers. Let $X = E^n(\rho_0) \times X_0$ and $X(\rho) = E^n(\rho) \times X_0$ for $\rho \leq \rho_0$. We see that X is an analytic manifold of pure dimension m . Let $\psi : X \rightarrow E^n(\rho_0)$ be the projection map. Now X is fibered by the fibers $\psi^{-1}(t) = X(t) = \{t\} \times X_0 \cong X_0$ for $t \in E^n(\rho_0)$. We take the sheaf S to be $S = (q\mathcal{C})_X$. With these notations we can state the following.

Theorem: The direct image sheaf $\psi_{(l)}((q\mathcal{C})_X)$ is a coherent sheaf of $\mathcal{C}_{E^n(\rho_0)}$ -modules for every $l \geq 0$.

Proof. Because X_0 is a compact analytic manifold we can find a finite Stein covering $\mathfrak{U} = \{ U_1 \dots U_{i^*} \}$ of X_0 . Let us put $\hat{U}_i = E^n(\rho_0) \times U_i$, then we see that $\hat{\mathfrak{U}} = \{ \hat{U}_1 \dots \hat{U}_{i^*} \}$ is a Stein covering of X . Let $\hat{\xi} = \{ \hat{\xi}_{i_0 \dots i_l} \} \in C^l(\hat{\mathfrak{U}}, (q\mathcal{C})_X)$. Now $\hat{\xi}_{i_0 \dots i_l}$ is a q -tuple of holomorphic functions on $E^n(\rho_0) \times U_{i_0 \dots i_l}$. Hence $\hat{\xi}_{i_0 \dots i_l}$ admits a Taylor series of the form $\hat{\xi}_{i_0 \dots i_l} = \sum_{|v|=0}^{\infty} \xi_{i_0 \dots i_l}^{(v)} (t/\rho_0)^v$ where $v = (v_1, \dots, v_n)$, $|v| = v_1 + \dots + v_n$ and $(t/\rho)^v = (t_1/\rho_1)^{v_1} \dots (t_n/\rho_n)^{v_n}$. The uniqueness of a Taylor series shows that $\{ \xi_{i_0 \dots i_l}^{(v)} \}$ is an alternating cochain over \mathfrak{U} . Putting $\xi_{(v)} = \{ \xi_{i_0 \dots i_l}^{(v)} \} \in C^l(\mathfrak{U}, (q\mathcal{C})_X)$ we may write $\hat{\xi} = \sum \xi_{(v)} (t/\rho)^v$. Introducing the map $(v) : \hat{\xi} \rightarrow \xi_{(v)}$ we get a commutative diagram of the form:

$$\begin{array}{ccc} C^l(\hat{\mathfrak{U}}, (q\mathcal{C})_X) & \xrightarrow{\delta} & C^{l+1}(\hat{\mathfrak{U}}, (q\mathcal{C})_X) \\ (v)\downarrow & & \downarrow(v) \\ C^l(\mathfrak{U}, (q\mathcal{C})_{X_0}) & \xrightarrow{\delta} & C^{l+1}(\mathfrak{U}, (q\mathcal{C})_{X_0}). \end{array}$$

We now need a *theorem of Cartan-Serre*: Let X_0 be a compact analytic manifold. Then, for any coherent analytic sheaf S the set $H^p(X_0, S)$ is a finite dimensional vector space for all $p \geq 0$.