THE COHERENCE OF DIRECT IMAGES

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THE COHERENCE OF DIRECT IMAGES

by H. GRAUERT

INTRODUCTION

The coherence of the direct images of coherent sheaves was treated in the paper [1]: H. Grauert: Ein Theorem der analytischen Garbentheorie und die Modulräume komplexer Strukturen (*Pub. Math.* IHES 1960, pp. 5-64, corrections 1963). This paper deals with the most general case and its technique is very difficult. The main point in the proof is the Hauptlemma on page 47. Here a proof of this Hauptlemma in the case of regular families of compact complex manifolds and locally free analytic sheaves is given. Although this special case is easier than the general, the ideas are practically the same. Therefore these lecture notes of some talks given by H. Grauert, Helsinki 1967, may lead to an understanding of the general proof. In these notes only the Hauptlemma is proved. The proof of coherence is omitted. This part is more formal and can be done like in [1] on p. 55. See [1] for applications of the theorem.

A detailed presentation of the proof in the general case is given also by Knorr [2].

COHOMOLOGY THEORY

In this paper we use Čech cohomology. We shall briefly show how this cohomology is defined. In the following discussion X denotes a connected complex analytic manifold, $\mathcal{O}$ is the sheaf of germs of holomorphic functions and S a sheaf of $\mathcal{O}$-modules. Let $\mathfrak{U} = \{ U_\iota \}_{\iota \in J}$ be an open covering of X. We put $U_{\iota_0 \dots \iota_l} = U_{\iota_0} \cap \dots \cap U_{\iota_l}$. We consider cochains of order l with values in S. Let us put $C^l(\mathfrak{U}, S) = \{ \xi \}$ where ξ denotes a full collection of crossections $\xi_{\iota_0 \dots \iota_l}$ over all $U_{\iota_0 \dots \iota_l}$. We always assume that $\xi_{\iota_0 \dots \iota_l}$ is anticommutative in its indices. In the system $\{ C^l(\mathfrak{U}, S) \}$ we have the usual coboundary map $\delta : C^l(\mathfrak{U}, S) \to C^{l+1}(\mathfrak{U}, S)$ which makes the system a complex. We put $Z^l(\mathfrak{U}, S) = \operatorname{Ker} \delta \subset C^l(\mathfrak{U}, S)$ and $B^l(\mathfrak{U}, S) = \delta(C^{l-1}(\mathfrak{U}, S))$. The $l-$ th cohomology group $H^l(\mathfrak{U}, S)$ with respect to the open covering $\mathfrak{U}$ is $Z^l(\mathfrak{U}, S)/B^l(\mathfrak{U}, S)$. An open covering $\mathfrak{V} = \{V_\nu\}_{\nu \in N}$ is finer than an open covering $\mathfrak{U} = \{U_\iota\}_{\iota \in J}$ if there exists an index map $\tau : N \to J$ such that $V_\nu \subset U_{\tau(\nu)}$ for $\nu \in N$. It follows that an element of $\Gamma(U_{\tau(\nu_0) \dots \tau(\nu_l)}, S)$ can be restricted to a continuous crossection over $V_{\nu_0 \dots \nu_l}$. In this way we get a map $\tau^* : C^l(\mathfrak{U}, S) \to C^l(\mathfrak{V}, S)$. The following diagram is commutative:

$$\begin{array}{ccc} C^l(\mathfrak{U}, S) & \xrightarrow{\tau^*} & C^l(\mathfrak{V}, S) \\ \delta\downarrow & & \delta\downarrow \\ C^{l+1}(\mathfrak{U}, S) & \xrightarrow{\tau^*} & C^{l+1}(\mathfrak{V}, S) \end{array}$$

It follows that we have a map $\tau^* : Z^l(\mathfrak{U}, S) \to Z^l(\mathfrak{V}, S)$. Let us put $Z^l(X, S) = \bigcup_{\mathfrak{U}} Z^l(\mathfrak{U}, S)$, where $\mathfrak{U}$ runs over all open coverings of X. In $Z^l(X, S)$ we can introduce an equivalence relation $\approx$ as follows: Let $\xi_1 \in Z^l(\mathfrak{U}, S)$ and $\xi_2 \in Z^l(\mathfrak{U}_1, S)$. We put $\xi_1 \approx \xi_2$ iff there exists $\mathfrak{U}_2$ such that $\mathfrak{U}_2$ is finer than $\mathfrak{U}$ and $\mathfrak{U}_1$ and $\xi_1|\mathfrak{U}_2 - \xi_2|\mathfrak{U}_2 \in B^l(\mathfrak{U}_2, S)$. Here we have put $\xi_\nu|\mathfrak{U}_2 = \tau_\nu^*(\xi_\nu)$ where τ_ν^* comes from an index map $\tau_\nu : \mathfrak{U}_2 \to \mathfrak{U}, \mathfrak{U}_1$. It is easy to check that the equivalence relation defined on $Z^l(X, S)$ is independent of the index maps. Now $H^l(X, S)$ is the set of equivalence classes in $Z^l(X, S)$. Because $C^l(\mathfrak{U}, S)$ is a module over the ring $I(X)$ of holomorphic functions on X it follows that $H^l(\mathfrak{U}, S)$ and $H^l(X, S)$ are modules over $I(X)$. We have a natural homomorphism $H^l(\mathfrak{U}, S) \to H^l(X, S)$. Let now $X' \subset X$ be an open subset. Then X' is a complex analytic manifold. We put $S' = S|X'$ and $\mathfrak{U}' = \mathfrak{U} \cap X' = \{U_\iota \cap X'\}$ and obtain an open covering of X'. The restriction of crossections gives a homomorphism $\gamma : C^l(\mathfrak{U}, S) \to C^l(\mathfrak{U}', S')$ which commutes with δ and any index map τ. Thus we obtain restriction homomorphisms: $H^l(\mathfrak{U}, S) \to H^l(\mathfrak{U}', S')$ and $H^l(X, S) \to H^l(X', S')$.

STEIN MANIFOLDS

A complex analytic manifold X is a Stein manifold if: 1) X is holomorphically convex, i.e. if $D = (x_\nu)_1^\infty$ is an infinite discrete set, then there exists $f \in I(X)$ such that $|f(D)| = \sup_\nu |f(x_\nu)|$ is infinite. 2) X can be spread holomorphically, i.e. for any $x \in X$ there exists $f_1 \dots f_N \in I(X)$ such that x is an isolated common zero of $f_1 \dots f_N$.

Let X be a complex analytic manifold. A Stein covering $\mathfrak{U} = \{ U_\iota \}_{\iota \in J}$ of X is an open covering of X such that every U_ι is Stein. We shall often use the following result:

Leray's Theorem: If $\mathfrak{U}$ is a Stein covering of X then $H^l(\mathfrak{U}, S) \to H^l(X, S)$ is an isomorphism for every coherent analytic sheaf S.

The isomorphism between $H^l(\mathfrak{U}, S)$ and $H^l(X, S)$ means the following: If $\underline{\xi} \in H^l(X, S)$ there exists $\xi \in Z^l(\mathfrak{U}, S)$ such that ξ maps into $\underline{\xi}$ under the natural homomorphism $Z^l(\mathfrak{U}, S) \to H^l(X, S)$ and moreover if $\xi \in Z^l(\mathfrak{U}, S)$ is mapped into zero in $H^l(X, S)$ there exist $\eta \in C^{l-1}(\mathfrak{U}, S)$ such that $\xi = \delta\eta$ in $C^l(\mathfrak{U}, S)$.

Direct images of sheaves

Let X and Y be complex analytic manifolds. Let $\psi : X \to Y$ be a holomorphic map and let S be an analytic sheaf on X. Now X is fibered by the fibers $X(y) = \psi^{-1}(y)$ for $y \in Y$. Let U be an open neighborhood of a point $y \in Y$, then $V = \psi^{-1}(U)$ is an open set in X. Hence V is a complex analytic manifold and the restriction of S to V gives an analytic sheaf on V. We can now define $H^l(V, S)$. Let us put $H^l_y = \bigcup\limits_{U} H^l(\psi^{-1}(U), S)$ where U runs over all open neighborhoods of y in Y. In H^l_y we introduce an equivalence relation as follows: $\xi_1 \in H^l(\psi^{-1}(U_1), S)$ and $\xi_2 \in H^l(\psi^{-1}(U_2), S)$ are equivalent iff there exists $U = U(y)$ in Y such that $U \subset U_1 \cap U_2$ and $\xi_1 | \psi^{-1}(U) = \xi_2 | \psi^{-1}(U)$ in $H^l(\psi^{-1}(U), S)$. We let $\psi_{(l)}(S)_{(y)}$ denote the set of equivalence classes in H^l_y. The equivalence class generated by $\xi \in H^l(\psi^{-1}(U), S)$ is denoted by ξ_y. The set $\psi_{(l)}(S)_{(y)}$ is called the set of germs of cohomology classes of dimension l along the fiber $X(y)$. Now $\psi_{(l)}(S)_{(y)}$ is an $\mathcal{O}_{y,Y}$-module. For if $g_y \in \mathcal{O}_{y,Y}$ we have a representative $g \in I(U)$ for some open neighborhood U of y. Then $g \circ \psi \in I(\psi^{-1}(U))$. If $\xi_y \in \psi_{(l)}(S)_{(y)}$ and U is sufficiently small we can find a representative $\xi \in H^l(\psi^{-1}(U), S)$ for ξ_y. Then we put $g_y \cdot \xi_y = ((g \circ \psi)\xi)_y$. Now we form $\psi_{(l)}(S) = \bigcup\limits_{y \in Y} \psi_{(l)}(S)_{(y)}$ where we introduce a sheaf topology. A base of the open sets are $\{ \xi_y : y \in U \}$ for $\xi \in H^l(\psi^{-1}(U), S)$. If $\xi \in H^l(X, S)$ then the map $y \to \xi_y$ is a cross-section in $\psi_{(l)}(S)$. We call it the direct image of ξ and denote it by $\psi_{(l)}(\xi)$. The sheaf $\psi_{(l)}(S)$ is the direct image sheaf of S of dimension l. Our main problem is to decide whether $\psi_{(l)}(S)$ is a coherent analytic sheaf of $\mathcal{O}_Y$-modules if S is a coherent analytic sheaf on X.

A VERY SPECIAL CASE

We shall consider a special case where our main problem is easily solved. Let X_0 be a compact analytic manifold of pure dimension $m - n$. We put $E^n(\rho_0) = \{ (t_1 \dots t_n) \in C^n \, ; \, |t_i| < \rho_i^0 \}$. Here $\rho_0 = (\rho_1^0 \dots \rho_n^0)$ is a fixed n-tuple of strictly positive numbers. Let $X = E^n(\rho_0) \times X_0$ and $X(\rho) = E^n(\rho) \times X_0$ for $\rho \leqslant \rho_0$. We see that X is an analytic manifold of pure dimension m. Let $\psi : X \to E^n(\rho_0)$ be the projection map. Now X is fibered by the fibers $\psi^{-1}(t) = X(t) = \{t\} \times X_0 \cong X_0$ for $t \in E^n(\rho_0)$. We take the sheaf S to be $S = (q\mathcal{O})_X$. With these notations we can state the following.

Theorem: The direct image sheaf $\psi_{(l)}((q\mathcal{O})_X)$ is a coherent sheaf of $\mathcal{O}_{E^n(\rho_0)}$-modules for every $l \geqslant 0$.

Proof. Because X_0 is a compact analytic manifold we can find a finite Stein covering $\mathfrak{U} = \{ U_1 \dots U_{\iota *} \}$ of X_0. Let us put $\hat{U}_\iota = E^n(\rho_0) \times U_\iota$, then we see that $\hat{\mathfrak{U}} = \{ \hat{U}_1 \dots \hat{U}_{\iota *} \}$ is a Stein covering of X. Let $\hat{\xi} = \{ \hat{\xi}_{\iota_0 \dots \iota_l} \} \in C^l(\hat{\mathfrak{U}}, (q\mathcal{O})_X)$. Now $\hat{\xi}_{\iota_0 \dots \iota_l}$ is a q-tuple of holomorphic functions on $E^n(\rho_0) \times U_{\iota_0 \dots \iota_l}$. Hence $\hat{\xi}_{\iota_0 \dots \iota_l}$ admits a Taylor series of the form $\hat{\xi}_{\iota_0 \dots \iota_l} = \sum_{|\nu|=0}^{\infty} \xi^{(\nu)}_{\iota_0 \dots \iota_l} (t/\rho_0)^\nu$ where $\nu = (\nu_1, \dots, \nu_n)$, $|\nu| = \nu_1 + \dots + \nu_n$ and $(t/\rho)^\nu = (t_1/\rho_1)^{\nu_1} \dots (t_n/\rho_n)^{\nu_n}$. The uniqueness of a Taylor series shows that $\{ \xi^{(\nu)}_{\iota_0 \dots \iota_l} \}$ is an alternating cochain over $\mathfrak{U}$. Putting $\xi_{(\nu)} = \{ \xi^{(\nu)}_{\iota_0 \dots \iota_l} \} \in C^l(\mathfrak{U}, (q\mathcal{O})_X)$ we may write $\hat{\xi} = \sum \xi_{(\nu)} (t/\rho)^\nu$. Introducing the map $(\nu) : \hat{\xi} \to \xi_{(\nu)}$ we get a commutative diagram of the form:

$$\begin{array}{ccc} C^l(\hat{\mathfrak{U}}, (q\mathcal{O})_X) & \xrightarrow{\delta} & C^{l+1}(\hat{\mathfrak{U}}, (q\mathcal{O})_X) \\ (\nu)\downarrow & & \downarrow(\nu) \\ C^l(\mathfrak{U}, (q\mathcal{O})_{X_0}) & \xrightarrow{\delta} & C^{l+1}(\mathfrak{U}, (q\mathcal{O})_{X_0}) . \end{array}$$

We now need a *theorem of Cartan-Serre:* Let X_0 be a compact analytic manifold. Then, for any coherent analytic sheaf S the set $H^p(X_0, S)$ is a finite dimensional vector space for all $p \geqslant 0$.

Using this theorem we conclude that $H^l(X_0, (q\mathcal{O})_{X_0})$ has a finite base $\mathfrak{b}_1 \ldots \mathfrak{b}_r$. By Leray's theorem we also have $H^l(\mathfrak{U}, (q\mathcal{O})_{X_0}) \cong H^l(X_0, (q\mathcal{O})_{X_0})$. Hence we can find $\mathfrak{b}_1 \ldots \mathfrak{b}_r \in Z^l(\mathfrak{U}, (q\mathcal{O})_{X_0})$ such that $\mathfrak{b}_\nu$ maps into $\mathfrak{b}_\nu$ under the natural homomorphism $Z^l(\mathfrak{U}, (q\mathcal{O})_{X_0}) \to H^l(X_0, (q\mathcal{O})_{X_0})$. We now introduce a pseudonorm in $C^l(\mathfrak{U}, (q\mathcal{O})_{X_0})$ as follows:

Norm definition. Let $\eta \in C^l(\mathfrak{U}, (q\mathcal{O})_{X_0})$. Then we put $\|\eta\| = \sup_{(\iota_0 \cdots \iota_l)} \|\eta_{\iota_0 \cdots \iota_l}\|$ and $\|\eta_{\iota_0 \cdots \iota_l}\| = \max_{1 \leq Q \leq q} \sup |\eta_Q(U_{\iota_0 \cdots \iota_l})|$, where, $\eta_{\iota_0 \cdots \iota_l} = (\eta_1, \ldots, \eta_q)$. Notice that it may happen that $\|\eta\| = +\infty$. Let $\mathfrak{V} = \{V_1 \ldots V_{\iota^*}\}$ be an open covering of X_0. The covering $\mathfrak{V}$ is much finer than $\mathfrak{U} = \{U_1 \ldots U_{\iota^*}\}$ if $V_\iota \subset\subset U_\iota$ holds for every ι. We write $\mathfrak{V} \ll \mathfrak{U}$ in that case. Let us now choose Stein coverings $\mathfrak{V}_1$ and $\mathfrak{V}$ such that $\mathfrak{V}_1 \ll \mathfrak{V} \ll \mathfrak{U}$. In $C^l(\mathfrak{V}, (q\mathcal{O})_{X_0})$ and $C^l(\mathfrak{V}_1, (q\mathcal{O})_{X_0})$ we introduce a pseudonorm just as in $C^l(\mathfrak{U}, (q\mathcal{O})_{X_0})$. If $\xi \in C^l(\mathfrak{U}, (q\mathcal{O})_{X_0})$ we have defined $\xi \mid \mathfrak{V} \in C^l(\mathfrak{V}, (q\mathcal{O})_{X_0})$. It follows that $\|\xi \mid \mathfrak{V}\| < \infty$ because $V_{\iota_0 \cdots \iota_l} \subset\subset U_{\iota_0 \cdots \iota_l}$. Let us now choose $\xi \in Z^l(\mathfrak{V}, (q\mathcal{O})_{X_0})$. Since $\mathfrak{b}_1 \ldots \mathfrak{b}_r \in Z^l(\mathfrak{U}, (q\mathcal{O})_{X_0})$ constitute a base of $H^l(X_0, (q\mathcal{O})_{X_0})$ it follows from Leray's theorem that $\xi = \sum a_\nu \mathfrak{b}_\nu | \mathfrak{V} + \delta\eta$ where $a_\nu \in C^1$ and $\eta \in C^{l-1}(\mathfrak{V}, (q\mathcal{O})_{X_0})$. Now we need the following.

Lemma: There exists a constant K such that $|a_\nu| \leqslant K\|\xi\|$ and $\|\eta \mid \mathfrak{V}_1\| \leqslant K\|\xi\|$.

The proof follows because by the Banach theorem the map $(a_1, \ldots, a_r, \eta) \to \xi$ of the Fréchet spaces $C^r \times C^{l-1}(\mathfrak{V}, q\mathcal{O}_{X_0})$ onto $Z^l(\mathfrak{V}, (q)\mathcal{O}_{X_0}))$ is open.

Let $\xi \in C^l(\mathfrak{U}, (q\mathcal{O})_{X_0})$. We can extend each $\xi_{\iota_0 \cdots \iota_l} \in q\,I(U_{\iota_0 \cdots \iota_l})$ constantly over $\hat{U}_{\iota_0 \cdots \iota_l} = E^n(\rho_0) \times U_{\iota_0 \cdots \iota_l}$. We get $\hat{\xi} \in Z^l(\hat{\mathfrak{U}}, (q\mathcal{O})_X)$ obtained from ξ by a constant extension. In particular we extend $\mathfrak{b}_1 \ldots \mathfrak{b}_r$ constantly to $\hat{\mathfrak{b}}_1 \ldots \hat{\mathfrak{b}}_r \in Z^l(\hat{\mathfrak{U}}, (q\mathcal{O})_X)$. Let $\hat{\mathfrak{b}}_1 \ldots \hat{\mathfrak{b}}_r$ be the images of $\hat{\mathfrak{b}}_1 \ldots \hat{\mathfrak{b}}_r$ in the direct image sheaf $\psi_{(l)}((q\mathcal{O})_X)$. Let now $\xi_0 \in \psi_{(l)}((q\mathcal{O})_X)_{(0)}$ where 0 is the origin of $E^n(\rho_0)$. By definition we can find $\xi \in H^l(X(\rho_1), q\mathcal{O})$ with $0 < \rho_1 \leqslant \rho_0$ which maps into ξ_0. Now $\hat{\mathfrak{U}}(\rho_1) = \{E^n(\rho_1) \times U_\iota\}$ is a Stein covering of $X(\rho_1)$. Hence Leray's theorem shows that we can find $\hat{\xi} \in Z^l(\hat{\mathfrak{U}}(\rho_1), (q\mathcal{O})_X)$ such that $\hat{\xi}$ maps into ξ_0. Let us write $\hat{\xi} = \sum \xi_{(\nu)}(t/\rho_1)^\nu$ where $\xi_{(\nu)} \in Z^l(\mathfrak{U}, (q\mathcal{O})_{X_0})$. Let us also choose $0 < \rho_2 < \rho_1$ and consider $\hat{\xi} \mid \hat{\mathfrak{V}}(\rho_2) = \hat{\xi}_1 \in Z^l(\hat{\mathfrak{V}}(\rho_2), (q\mathcal{O})_X)$. Let us write $\hat{\xi}_1 = \sum \xi^*_{(\nu)}(t/\rho_2)^\nu$. Obviously we get $\xi^*_{(\nu)} = (\rho_2/\rho_1)^\nu \xi_{(\nu)} \mid \mathfrak{V}$. It follows easily that $\sup_\nu \|\xi^*_{(r)}\| < \infty$.

The previous lemma shows that $\overset{*}{\xi}_{(\nu)} = \sum a_{\nu\lambda}\, \mathfrak{b}_\lambda + \delta\eta_\nu$ where $\eta_\nu \in C^{l-1}(\mathfrak{V})$ with $\| \eta_\nu \mid \mathfrak{V}_1 \| \leqslant K \| \overset{*}{\xi}_{(\nu)} \|$ and $| a_{\nu\lambda} | \leqslant K \| \overset{*}{\xi}_{(\nu)} \|$. Let us put $a_\lambda = \sum a_{\nu\lambda} (t/\rho_2)^\nu$ and $\hat{\eta} = \sum \eta_\nu (t/\rho_2)^\nu$. We see that $\hat{\eta} \in C^{l-1}(\hat{\mathfrak{V}}_1(\rho_2))$ and $a_\lambda \in I(E^n(\rho_2))$. An easy computation gives $\hat{\xi}_1 \mid \hat{\mathfrak{V}}_1(\rho_2) = \sum a_\lambda \hat{\mathfrak{b}}_\lambda \mid \hat{\mathfrak{V}}_1(\rho_2) + \delta\, \eta$. It follows by definition that $\xi_0 = \sum a_\lambda \hat{\mathfrak{b}}_\lambda$. We have now proved that $\hat{\mathfrak{b}}_1 \dots \hat{\mathfrak{b}}_r$ generate $\psi_{(l)}((q\mathcal{O})_X)$ at the origin. It follows in the same way that $\hat{\mathfrak{b}}_1 \dots \hat{\mathfrak{b}}_r$ generate $\psi_{(l)}((q\mathcal{O})_X)$ for every $t \in E^n(\rho_0)$ because it is enough to do everything in a polydisc around t. Now we also prove that the sheaf $\psi_{(l)}((q\mathcal{O})_X)$ is free, i.e. there are no relations between $\hat{\mathfrak{b}}_1 \dots \hat{\mathfrak{b}}_r$ at any point. Say for example that $a_1 \hat{\mathfrak{b}}_1 + \dots + a_r \hat{\mathfrak{b}}_r = 0$ at $\psi_{(l)}((q\mathcal{O})_X)_{(0)}$ where a_ι are germs of analytic functions at the origin in $E^n(\rho_0)$. Hence $\tilde{a}_1 \hat{\mathfrak{b}}_1 + \dots + \tilde{a}_r \hat{\mathfrak{b}}_r = 0$ in $H^l(X(\rho), (q\mathcal{O})_X)$ for some $\rho > 0$ with $\tilde{a}_\iota \in I(E^n(\rho))$. It follows that $\sum \tilde{a}_\nu \hat{\mathfrak{b}}_\nu = \delta\, \hat{\xi}$ in $X(\rho)$ for some $\hat{\xi} \in C^{l-1}(\hat{\mathfrak{U}}(\rho), (q\mathcal{O})_X)$. Take a point $t \in E^n(\rho)$ where some $\tilde{a}_\nu \neq 0$. Now we see that on $\{ t \} \times X_0$ we have $\tilde{a}_1(t)\, \mathfrak{b}_1 + \dots + \tilde{a}_r(t)\, \mathfrak{b}_r = \partial\, \hat{\xi} \mid \{ t \} \times X_0 \in C^{l-1}(\mathfrak{U}, (q\mathcal{O})_{X_0})$. This gives a contradiction to the fact that $\mathfrak{b}_1 \dots \mathfrak{b}_r$ are a base of $H^l(X_0, (q\mathcal{O})_{X_0})$.

Measure charts

Let X be a connected complex analytic manifold of dimension m. Let F be a holomorphic vector bundle of rank q on X and $\mathbf{F}$ the sheaf of holomorphic crossections in F. This sheaf is locally free. A regular proper holomorphic map $\psi\colon X \to E^n$ is given. Let us put $X_0 = \psi^{-1}(0)$. Now X_0 is a compact analytic manifold of dimension $m - n$. We now introduce special open coverings around X_0 in X.

Definition. A measure chart $\mathcal{W} = (\hat{W}, \Phi, \Theta, \rho)$ is a quadruple satisfying the conditions:

1) $\hat{W} \subset X$ is open and $W = \hat{W} \cap X_0$ is Stein.

2) $\Phi\colon \hat{W} \to E^n(\rho) \times W$ is a biholomorphic map such that the following diagram is commutative:

$$\begin{array}{ccc} \hat{W} & \xrightarrow{\Phi} & E^n(\rho) \times W \\ & \psi \searrow \quad \swarrow \pi & \\ & E^n(\rho)\,. & \end{array}$$

Here π is the projection map.

3) $\Theta\colon F|\hat{W} \to \hat{W} \times \mathbf{C}^q$ is a trivialization of F on $\hat{W}$.

If $\mathscr{W}$ is a given measure chart on X we can identify the sheaf $(\hat{W}, \mathbf{F}|\hat{W})$ of $\mathscr{O}_X$-modules with the sheaf $(W \times E^n(\rho), q\,\mathscr{O})$ using Φ and Θ. If $U \subset W$ is open and $\rho' \leqslant \rho$ we put $\hat{U}(\rho') = \Phi^{-1}(U \times E^n(\rho'))$. Hence if $\hat{s} \in \Gamma(\hat{U}(\rho'), F)$ we can identify $\hat{s}$ with an element of $\Gamma(U \times E^n(\rho'), q\,\mathscr{O})$. We shall simply denote this element of $\Gamma(U \times E^n(\rho'), q\,\mathscr{O})$ by the same letter $\hat{s}$. Now we can expand $\hat{s}$ in a Taylor series: $\hat{s} = \sum_{|\nu|=0}^{\infty} s_\nu (t/\rho')^\nu$ where $s_\nu \in qI(U)$.

Definition of a norm. When $\hat{s} \in \Gamma(\hat{U}(\rho'), F)$ we put $\|\hat{s}\| = \sup_\nu |s_\nu(U)|$.

Strictly speaking the norm $\|\hat{s}\|$ is taken with respect to the measure chart $\mathscr{W}$.

It is not hard to see that for every point $x \in X_0$ there exists a measure chart $\mathscr{W}$ such that $x \in \hat{W}$. In particular we can cover X_0 by finitely many measure charts $\mathscr{W}_\iota = (\hat{W}_\iota, \Phi_\iota, \Theta_\iota, \rho_\iota)$, i.e. $X_0 \subset\subset \bigcup_1^{\iota^*} \hat{W}_\iota$. We remark that it follows that $X(\rho) = \psi^{-1}(E^n(\rho)) \subset\subset \bigcup_1^{\iota^*} \hat{W}_\iota$ for some $\rho > 0$ with $\rho \leqslant \rho_\iota$ because ψ is a proper map. The collection $\mathscr{W} = \{\mathscr{W}_\iota\}_1^{\iota^*}$ is called an atlas around X_0. From now on $\mathscr{W}$ is a fixed atlas.

Measure coverings. We shall define measure coverings with respect to the given atlas $\mathscr{W}$ above. If $U \subset W_\iota$ is open we put $(U)_\iota(\rho) = \Phi_\iota^{-1}(U \times E^n(\rho))$ when $\rho \leqslant \rho_\iota$. We see that $(U)_\iota(\rho) \subset \hat{W}_\iota$ and $(U)_\iota(\rho)$ is Stein if U is Stein. Let $\mathfrak{U} = \{U_\iota\}_1^{\iota^*}$ be a Stein covering of X_0 with $U_\iota \subset\subset W_\iota$ for each ι. Let $\rho > 0$ with $\rho < \min_\iota \rho_\iota$. We put $\hat{U}_\iota(\rho) = (U_\iota)_\iota(\rho)$. We see that $\hat{U}_\iota(\rho) \subset\subset \hat{W}_\iota$ and $\hat{U}_\iota(\rho)$ are Stein. It is now required that $\hat{\mathfrak{U}}(\rho) =$

$= \{ \hat{U}_\iota(\rho) \}_1^{\iota^*}$ is a Stein covering of $X(\rho)$. We say then that $\hat{\mathfrak{U}}(\rho)$ is a measure covering of $X(\rho)$.

Admissible refinements of measure coverings. Let $\hat{\mathfrak{U}}(\rho)$ and $\hat{\mathfrak{U}}^*(\rho)$ be two measure coverings of $X(\rho)$. We say that $\hat{\mathfrak{U}}^*(\rho)$ is an admissible refinement of $\hat{\mathfrak{U}}(\rho)$ if the following conditions hold:

1) $U^*_\iota \subset \subset U_\iota$ for each ι.

2) If $U^*_{\iota_0 \dots \iota_\lambda} = U^*_{\iota_0} \cap \dots \cap U^*_{\iota_\lambda}$ we put $(U^*_{\iota_0 \dots \iota_\lambda})_\nu = \Phi_\nu^{-1}(U^*_{\iota_0 \dots \iota_\lambda} \times E^n(\rho))$ for each $\nu \in \{ \iota_0 \dots \iota_\lambda \}$. It is now required that $(U^*_{\iota_0 \dots \iota_\lambda})_\nu \subset (U_{\iota_0 \dots \iota_\lambda})_\mu$ for all $\nu, \mu \in \{ \iota_0 \dots \iota_\lambda \}$.

3) $\hat{U}^*_{\iota_0 \dots \iota_\lambda} = \hat{U}^*_{\iota_0} \cap \dots \cap \hat{U}^*_{\iota_\lambda} \subset (U_{\iota_0 \dots \iota_\lambda})_\mu$ for each $\mu \in \{ \iota_0 \dots \iota_\lambda \}$.

Existence of admissible refinements of measure coverings

Existence Theorem. For every fixed integer s we can find, for some $\rho > 0$, a sequence $\mathfrak{U}_s \ll \mathfrak{U}_{s-1} \ll \dots \ll \mathfrak{U}_1 \ll \mathfrak{U}_0$ of finer measure coverings of $X(\rho)$ each of which is an admissible refinement of the following.

Proof. We first construct a measure covering of $X(\rho)$ for some $\rho < \min \rho_\iota$. Let $\mathfrak{U}_0 = \{ \mathfrak{U}_\iota \}_1^{\iota^*}$ be a Stein covering of X_0 such that $U_\iota \subset \subset W_\iota$ for $\iota \in \{ 1, \dots, \iota^* \}$. Choose a fixed $\rho_0 < \min \rho_\iota$. Now the open sets $\Phi_\iota^{-1}(U_\iota \times E^n(\rho_0))$ cover X_0 and hence they also cover $X(\rho)$ for some sufficiently small ρ. Hence $\mathfrak{U}_0$ defines a measure covering of $X(\rho)$. It is also clear that $\mathfrak{U}_0$ defines a measure covering of $X(\rho')$ for each $\rho' \leqslant \rho$. Let us now construct $\mathfrak{U}_1$. We let $\mathfrak{U}^* = \{ U^*_\iota \}_1^{\iota^*}$ be a Stein covering such that $U^*_\iota \subset \subset U_\iota$ always holds. Now we can find $\rho_1 \leqslant \rho$ such that $\{ \hat{U}^*_\iota(\rho_1) = \Phi_\iota^{-1}(U^*_\iota \times E^n(\rho_1)) \}_1^{\iota^*}$ cover $X(\rho_1)$. Hence $\hat{\mathfrak{U}}^*(\rho_1)$ and $\hat{\mathfrak{U}}(\rho_1)$ are measure coverings of $X(\rho_1)$. But we do not yet know if $\hat{\mathfrak{U}}^*(\rho_1) \ll \hat{\mathfrak{U}}(\rho_1)$. We claim that if $\rho_2 \leqslant \rho_1$ is sufficiently small then $\hat{\mathfrak{U}}^*(\rho_2) \ll \hat{\mathfrak{U}}(\rho_2)$. For suppose this is false. Say that 2) fails for $\hat{\mathfrak{U}}^*(\rho_2)$ and $\hat{\mathfrak{U}}(\rho_2)$ when $0 < \rho_2 \leqslant \rho_1$. Hence $\Phi_\nu^{-1}(U^*_{\iota_0 \dots \iota_\lambda} \times E^n(\rho_2)) - \Phi_\mu^{-1}(U_{\iota_0 \dots \iota_\lambda} \times E^n(\rho_2))$ are non empty for suitable indices while $\rho_2 \to 0$. Choose a point x_t from each of these sets. Because $x_t \in X(\rho_1)$ which is relatively compact we may assume that $x_t \to x_0$. Obviously we get $x_0 \in \overline{U^*_{\iota_0 \dots \iota_\lambda}} - U_{\iota_0 \dots \iota_\lambda}$, a contradiction because $\overline{U^*_{\iota_0\cdots\iota_\lambda}} \subset \overline{U^*_{\iota_0}} \cap \ldots \cap \overline{U^*_{\iota_\lambda}} \subset U_{\iota_0\cdots\iota_\lambda}$. In the same way we can prove that condition 3) is satisfied if ρ_2 is sufficiently small and the theorem is clear.

General Theory

Let G be an analytic manifold. We put $\hat{G} = G \times E^n(\rho_1)$ where ρ_1 is an n-tuple of positive numbers. Let $\pi\colon \hat{G} \to E^n(\rho_1)$ and $\mathfrak{P}\colon \hat{G} \to G$ be the projection maps. $\hat{G}^* \subset \hat{G}$ denotes an open subset and $G^* = \hat{G}^* \cap G \times \{0\}$.

The set G^* can be identified with an open subset of G. We denote by $\alpha\colon G^* \times E^n(\rho_1) \to \hat{G}^*$ a biholomorphic fiber preserving map, i.e. $\pi \circ \alpha = \pi^*$ where $\pi^*\colon G^* \times E^n(\rho_1) \to E^n(\rho_1)$ is the natural projection. Let $\rho \leqslant \rho_2 = \gamma\rho_1 < \rho_1$ where $0 < \gamma < 1$ is a fixed number. We put $\hat{G}(\rho) = G \times E^n(\rho)$. If f is a holomorphic function on $\hat{G}(\rho)$ we write $f = \sum a_\nu (t/\rho)^\nu$ with $a_\nu \in I(G)$. We define the norm $\|f\|_\rho$ of f by $\|f\|_\rho = \sup\limits_\nu \{\sup |a_\nu(G)|\}$. If $f \in I(\hat{G}(\rho))$ we see that $f \circ \alpha$ is a well defined function on $G^* \times E^n(\rho)$ because α is fiber preserving. We define $\|f \circ \alpha\|_\rho$ using G^* instead of G as above. We have the proposition:

Proposition 1. There exists a constant K such that $\|f \circ \alpha\|_\rho \leqslant K\|f\|_\rho$ where $K = K(\rho_2)$ is independent of $\rho \leqslant \rho_2$.

Proof. We write $f = \sum\limits_{|\nu|=0}^{\infty} a_\nu (t/\rho)^\nu$ with $a_\nu \in I(G)$. Now we get $f \circ \alpha = \sum (a_\nu \circ \mathfrak{P} \circ \alpha)(t/\rho)^\nu$ because α is fiber preserving. Since $\mathfrak{P}(\hat{G}^*) \subset G$ we get $|a_\nu \circ \mathfrak{P}(\hat{G}^*)| \leqslant |a_\nu(G)| \leqslant \|f\|_\rho$. Now $a_\nu \circ \mathfrak{P} \circ \alpha$ admits a Taylor series: $a_\nu \circ \mathfrak{P} \circ \alpha = \sum C_{\nu\lambda}(t/\rho)^\lambda$ with $C_{\nu\lambda} \in I(G^*)$. Since $|\sum C_{\nu\lambda}(t/\rho)^\lambda| \leqslant \|f\|_\rho$ in $G^* \times E^n(\rho_1)$ and $\rho \leqslant \rho_2 = \gamma\rho_1$ Cauchy's inequalities give us $|C_{\nu\lambda}(G^*)| \leqslant \|f\|_\rho \gamma^{|\lambda|}$. Let us put $b_\mu = \sum\limits_{\nu+\lambda=\mu} C_{\nu\lambda}$. We get $|b_\mu(G^*)| \leqslant \|f\|_\rho \sum \gamma^{|\lambda|} = \|f\|_\rho (1-\gamma)^{-n} = K\|f\|_\rho$. Now we can write $f \circ \alpha = \sum\limits_\nu a_\nu \circ \mathfrak{P} \circ \alpha (t/\rho)^\nu = \sum\limits_{\lambda,\nu} C_{\nu\lambda}(t/\rho)^\lambda (t/\rho)^\nu = \sum\limits_\mu b_\mu (t/\rho)^\mu$. By definition we have $\|f \circ \alpha\|_\rho = \sup\limits_\mu |b_\mu(G^*)| \leqslant K\|f\|_\rho$.

Let us now consider $\mathbf{h} = (h_{\nu\mu})$ which is a $q \times q$ matrix with $h_{\nu\mu} \in I(\hat{G})$. The $h_{\nu\mu}$ are also assumed to be bounded on $\hat{G}$.

Proposition 2. Let $\mathbf{f} = (f_1 \dots f_q) \in qI(\hat{G}(\rho))$. Then $\| \mathbf{h}(\mathbf{f}) \|_\rho \leqslant K \| \mathbf{f} \|_\rho$. As before $\rho \leqslant \rho_2 = \gamma \rho_1 < \rho_1$ and K only depends on ρ_2.

Proof. We have $\mathbf{h}(\mathbf{f}) = (g_1 \dots g_q)$ with $g_\nu = \sum_\mu h_{\nu\mu} f_\mu$. Let us write $h_{\nu\mu} = \sum_\lambda a_{\nu\mu\lambda} (t/\rho)^\lambda$. By assumption $| h_{\nu\mu}(\hat{G}) | \leqslant M$ for some constant M and hence we have, by Cauchy's inequalities, $| a_{\nu\mu\lambda}(G) | \leqslant M\gamma^{|\lambda|}$. Let us also write $f_\mu = \sum_\lambda b_{\mu\lambda} (t/\rho)^\lambda$. By definition $\sup_{\mu,\lambda} | b_{\mu\lambda}(G) | = \| \mathbf{f} \|_\rho$. Now we get $g_\nu = \sum_\mu \sum_{\lambda_1,\lambda_2} a_{\nu\mu\lambda_1} b_{\mu\lambda_2} (t/\rho)^{\lambda_1+\lambda_2} = \sum C_{\nu\lambda}(t/\rho)^\lambda$ where $C_{\nu\lambda} = \sum_\mu \sum_{\lambda_1+\lambda_2=\lambda} a_{\nu\mu\lambda_1} b_{\mu\lambda_2}$. We get easily $| C_{\nu\lambda}(G) | \leqslant qM \| \mathbf{f} \|_\rho (1-\gamma)^{-n} = K \| \mathbf{f} \|_\rho$. Hence $\| \mathbf{h}(\mathbf{f}) \|_\rho = \sup_\nu \| g_\nu \|_\rho = \sup_{\nu,\lambda} | C_{\nu\lambda}(G) | \leqslant K \| \mathbf{f} \|_\rho$.

We shall now apply these two propositions to our situation. Let $G^* \subset G \subset W_{\iota_0 \dots \iota_\lambda} \subset X_0$. Here G^* and G are open sets and $W_{\iota_0 \dots \iota_\lambda}$ comes from the measure atlas $\mathscr{W}$. As before $\rho \leqslant \rho_2 < \rho_* = \min \rho_\iota$. We are given ι and ι' from $\{ \iota_0, \dots, \iota_\lambda \}$ and the following inclusions are assumed: $(G^*)_{\iota'}(\rho_1) \subset (G)_\iota(\rho_1)$, $(G^*)_{\iota'}(\rho_1) \subset\subset \hat{W}_{\iota'}$, $(G)_\iota(\rho_1) \subset\subset \hat{W}_\iota$.

The following theorem is very important.

Theorem I. Let $S \in \Gamma((G)_\iota(\rho), \mathrm{F})$. Then $\| S | (G^*)_{\iota'}(\rho) \|_{\iota'} \leqslant K \| S \|_\iota$. K depends only on ρ_2.

Proof. We have the following diagram:

$$\begin{array}{ccc} (G)_\iota(\rho_1) & \xrightarrow{\Phi_\iota} & G \times E^n(\rho_1) \\ \text{injection} \uparrow & & \uparrow \alpha \\ (G^*)_{\iota'}(\rho_1) & \xrightarrow{\Phi_{\iota'}} & G^* \times E^n(\rho_1) \end{array}$$

α being a fiber preserving holomorphic map. We identify $S | (G^*)_{\iota'}(\rho)$ with an element of $qI(G^* \times E^n(\rho))$ using the trivialization of F in the chart $\mathscr{W}_{\iota'}$. Call this element S^*. Also S itself is considered as an element of $qI(G \times E^n(\rho))$ using the trivialization in the chart $\mathscr{W}_\iota$. Now we have $S^* = \mathbf{h}(S \circ \alpha)$ where $\mathbf{h}$ is a $q \times q$ matrix. The elements of $\mathbf{h}$ are holomorphic functions defined on $\Phi_{\iota'}(\hat{W}_{\iota\iota'}) \supset\supset G^* \times E^n(\rho_1)$. Hence the elements of $\mathbf{h}$ are bounded on $G^* \times E^n(\rho_1)$. It is now obvious how we can use 1) and 2) to finish the proof.

We shall need one more general result. Let G be an analytic manifold. G is assumed to be Stein and $R^* = \{ U_1, \dots, U_{\iota^*} \}$ a Stein covering of G.

The set $G^* \subset G$ is open and $R^{**} = \{ V_1, ..., V_{\iota^*} \}$ an open covering of G^* such that $V_\iota \subset\subset U_\iota$ for $\iota \in \{ 1, ..., \iota^* \}$. We have:

Cartan's Theorem. There exists a constant K such that if $\xi \in Z^l(R^*, q\mathcal{O})$ then $\xi \mid R^{**} = \delta\eta$ where $\eta \in C^{l-1}(R^{**}, q\mathcal{O})$ and $\| \eta \| \leqslant K \| \xi \|$ for $l \geqslant 1$.

This is a simple consequence of Theorem B and Banach's open mapping theorem.

Now we apply Cartan's theorem. We keep the notations as above. Let $\hat{G} = G \times E^n(\rho)$ and put $\hat{R}^* = \{ U_\iota \times E^n(\rho) \}$. Now $\hat{R}^*$ is a Stein covering of $\hat{G}$. Let $\hat{G}^* = G^* \times E^n(\rho)$ and $\hat{R}^{**} = \{ V_\iota \times E^n(\rho) \}$. Let $\hat{\xi} \in Z^l(\hat{R}^*, q\mathcal{O})$ and write $\hat{\xi} = \sum \xi_{(\nu)} (t/\rho)^\nu$ with $\xi_{(\nu)} \in Z^l(R^*, q\mathcal{O})$. We assume $\| \hat{\xi} \|_\rho = \sup_\nu \| \xi_{(\nu)} \| < \infty$. Now Cartan's theorem gives $\xi_{(\nu)} \mid R^{**} = \delta\eta_\nu$ with $\eta_\nu \in C^{l-1}(R^{**}, q\mathcal{O})$ and $\| \eta_\nu \| \leqslant K \| \xi_{(\nu)} \| < \infty$. It follows that $\hat{\eta} = \sum \eta_\nu (t/\rho)^\nu$ is well defined in $C^{l-1}(\hat{R}^{**}, q\mathcal{O})$ and by definition we have $\| \hat{\eta} \|_\rho \leqslant K \| \hat{\xi} \|_\rho$.

Smoothing

We are given a sequence of admissible refinements of measure coverings in $X(\rho_1)$. Here $\rho_1 < \rho_0 = \min \rho_\iota$ as usual. Let l be a fixed integer $\geqslant 1$. We are given $\mathfrak{V}^* \ll \mathfrak{V}' = \mathfrak{V}_{3l} \ll \mathfrak{V}_{3l-1} \ll ... \ll \mathfrak{V}_1 \ll \mathfrak{V}_0 \ll \mathfrak{V} \ll \mathfrak{U}^* \ll \mathfrak{U} = \mathfrak{U}_{3l} \ll ... \ll \mathfrak{U}_0 \ll \mathfrak{U}'$. Here it is also required that $(\mathfrak{V}_{\nu+1}, \mathfrak{U}_{\nu+1}) \ll (\mathfrak{V}_\nu, \mathfrak{U}_\nu)$; $(\mathfrak{V}^*, \mathfrak{U}^*) \ll (\mathfrak{V}', \mathfrak{U})$ and $(\mathfrak{V}_0, \mathfrak{U}_0) \ll (\mathfrak{V}, \mathfrak{U}')$. These extra conditions mean: 1) $\hat{U}^{(\nu+1)}_{i_0 \dots i_k} \cap \hat{V}^{(\nu+1)}_{\iota_0 \dots \iota_l} \subset (U^{(\nu)}_{i_0 \dots i_k} \cap V^{(\nu)}_{\iota_0 \dots \iota_l})_i$ for each $i \in \{ i_0, ..., i_k \}$ and 2) $(U^{(\nu+1)}_{i_0 \dots i_k} \cap V^{(\nu+1)}_{\iota_0 \dots \iota_l})_j \subset (U^{(\nu)}_{i_0 \dots i_k} \cap V^{(\nu)}_{\iota_0 \dots \iota_l})_i$ for all $i, j \in \{ i_0, \dots i_k, \iota_0, \dots \iota_l \}$. Recall that all operations are done with respect to ρ_1. Let us put $\hat{R}^{(\nu)}_{i_0 \dots i_k \iota_0 \dots \iota_\kappa} = \hat{U}^{(\nu)}_{i_0 \dots i_k} \cap \hat{V}^{(\nu)}_{\iota_0 \dots \iota_\kappa}$. We consider elements $\xi_{i_0 \dots i_k \iota_0 \dots \iota_\kappa} \in \hat{\Gamma}(\hat{R}^{(\nu)}_{i_0 \dots i_k \iota_0 \dots \iota_\kappa}, \mathbf{F})$. Now we take a full collection $\hat{\xi} = \{ \hat{\xi}_{i_0 \dots i_k \iota_0 \dots \iota_\kappa} \}$ of such elements which is anticommutative in $\{ i_0, \dots i_k \}$ and $\{ \iota_0, ..., \iota_\kappa \}$. In this way we get a double complex $C_\nu^{k,\kappa}$. Here $\delta : C_\nu^{k,\kappa} \to C_\nu^{k+1,\kappa}$ and $\partial : C_\nu^{k,\kappa} \to C_\nu^{k,\kappa+1}$ are the usual coboundary operators.

Norm in $C_\nu^{k,\kappa}$: Let $\hat{\xi} \in C_\nu^{k,\kappa}$; we put

$\| \hat{\xi} \|_\rho = \max\limits_{i,(i_0,\ldots i_k,\iota_0,\ldots,\iota_\kappa)} \{ \| \hat{\xi}_{i_0\ldots i_k \iota_0\cdots\iota_\kappa} | (R^{(\nu+1)}_{i_0\ldots i_k \iota_0\cdots\iota_\kappa})_i(\rho) \|_i$ with $i \in \{i_0,\ldots, i_k\}\}$. Here $\rho \geqslant \rho_1$ and $R^{(\nu+1)}_{i_0\ldots i_k, \iota_0\ldots\iota_\kappa} = U^{(\nu+1)}_{i_0\ldots i_k} \cap V^{(\nu+1)}_{\iota_0\ldots\iota_\kappa}$ and $\| \, \|_i$ is taken with respect to the chart $\mathscr{W}_i$ as usual.

SMOOTHING LEMMA: Let $\kappa > 0$. There exists a constant K such that: If $\hat{\xi} \in C_\nu^{k,\kappa}$ with $\partial\hat{\xi} = 0$ and $\| \hat{\xi} \|_\rho < \infty$ then we can find $\hat{\eta} \in C^{k,\kappa-1}_{\nu+3}$ such that $\hat{\xi} \mid C^{k,\kappa}_{\nu+3} = \partial\hat{\eta}$ and $\| \hat{\eta} \|_\rho \leqslant K \| \hat{\xi} \|_\rho$. Here $\rho \leqslant \rho_2 = \gamma\rho_1$ with $0 < \gamma < 1$ and K depends only on ρ_2.

Proof. Let us fix $i_0, \ldots, i_k$ in the following discussion. Let $G = U^{(\nu+1)}_{i_0\ldots i_k}$ and put $\hat{G} = (G)_i(\rho_1)$ for some $i \in \{ i_0, \ldots, i_k \}$ which is also fixed now. Now G is Stein in X_0 and $\hat{G}$ is Stein in X. We put $R^* = G \cap \mathfrak{V}_{\nu+1}$ which is a Stein covering of G. Also $\hat{R}^* = \{ (G \cap V_\iota^{(\nu+1)})_i(\rho_1) \}_{\iota=1,\ldots,\iota^*}$ is a Stein covering of $\hat{G}$. Let $\hat{\xi} = \{ \hat{\xi}_{i_0,\ldots i_k,\iota_0\ldots\iota_\kappa} \}$. Now we look at the elements of $\{ \hat{\xi}_{i_0,\ldots i_k,\iota_0\ldots\iota_\kappa} \} = \hat{\xi}_{i_0,\ldots i_k} \in Z^\kappa(\hat{R}^*, \mathbf{F})$. Here $i_0, \ldots i_k$ is fixed as above. We get a cocycle because we have assumed that $\partial\hat{\xi} = 0$. More precisely we have considered the restriction of $\hat{\xi}_{i_0,\ldots i_k, \iota_0,\ldots,\iota_\kappa}$ to $\hat{R}^*$. We must verify that this restriction is possible.

Verification: By definition of $Z^\kappa(\hat{R}^*, \mathbf{F})$ we have to look at sets of the following type: (these are the sets where the cross-sections are defined) $(G \cap V^{(\nu+1)}_{\iota_0})_i \cap \ldots \cap (G \cap V^{(\nu+1)}_{\iota_\kappa})_i = (G \cap V^{(\nu+1)}_{\iota_0\ldots\iota_\kappa})_i = (R^{(\nu+1)}_{i_0\ldots i_k \iota_0\cdots\iota_\kappa})_i$. Now by 2) we have $(R^{(\nu+1)}_{i_0\ldots i_k\iota_0\cdots\iota_\kappa})_i \subset \bigcap\limits_j (R^{(\nu)}_{i_0\cdots i_k\iota_0\cdots\iota_\kappa})_j \subset (U^{(\nu)}_{i_0})_{i_0} \cap \ldots \cap (V^{(\nu)}_{\iota_\kappa})_{\iota_\kappa} = \hat{R}^{(\nu)}_{i_0\cdots i_k\iota_0\cdots\iota_\kappa}$. Q.E.D.

Now we put $G^* = U^{(\nu+2)}_{i_0\ldots i_k} \subset\subset G$. We let $\hat{R}^{**} = \{(G^* \cap V_\iota^{(\nu+2)})_i\}_{\iota=1,\ldots,\iota^*}$. The system $\hat{R}^{**}$ is a Stein covering of $(G^*)_i$. We are in a good position now. For we are given $\hat{\xi}_{i_0,\ldots i_k} \in Z^\kappa(\hat{R}^*, \mathbf{F})$. Here $\hat{R}^*$ is a Stein covering of $\hat{G}$ and $\hat{G}$ is a Stein manifold. We are working in the chart $\mathscr{W}_i$ where the usual identifications are used. Hence we arrive at the following situation: G is a Stein manifold with a Stein covering $R^* = \mathfrak{V}_{\nu+1} \cap G$. Also $G^* \subset\subset G$ and $R^{**} = \mathfrak{V}_{\nu+2} \cap G^*$ is a Stein covering of G^* such that $R^{**} \subset\subset R^*$. The cocycle $\hat{\xi}_{i_0,\ldots i_k}$ is now considered as an element of $Z^\kappa(\hat{R}^*, q\mathcal{O})$ which we simply call $\hat{\xi}_{i_0 \dots i_k}$ again. Now we apply the result after Cartan's theorem. Hence we can find a constant K such that for every $\rho \leqslant \rho_2$ we get $\eta \in \in C^{\kappa-1}(\hat{R}^{**}, q\mathcal{O})$ and $\| \eta \|_\rho \leqslant K \| \hat{\xi}_{i_0,\dots i_k} \|_\rho$ with $\partial\eta = \hat{\xi}_{i_0 \dots i_k}$. But this means precisely that we can find $\hat{\eta}_{i_0,\dots i_k} \in C^{\kappa-1}(\hat{R}^{**}(\rho), \mathbf{F})$ such that $\| \hat{\eta}_{i_0 \dots i_k} \|_{i,\rho} \leqslant K \| \hat{\xi}_{i_0 \dots i_k} \|_{i,\rho}$ with $\hat{\xi}_{i_0 \dots i_k} = \partial \hat{\eta}_{i_0 \dots i_k}$. We have only constructed $\hat{\eta}_{i_0 \dots i_k}$ using a fixed $i \in \{ i_0, \dots, i_k \}$. Now we must let $(i_0, \dots, i_k)$ vary. For each $(i_0, \dots i_k)$ we choose some i which only depends on the unordered $(k+1)$-tupel $(i_0, \dots, i_k)$ and construct an element $\hat{\eta}_{i_0,\dots i_k}$ as above. Now we can restrict everything to $C^{k,\kappa-1}_{\nu+3}$.

Verification: Consider a set where cross-sections over $C^{k,\kappa-1}_{\nu+3}$ have to be defined, i.e. a set $\hat{U}^{(\nu+3)}_{i_0 \dots i_k} \cap \hat{V}^{(\nu+3)}_{\iota_0 \dots \iota_\kappa}$. But by 1) follows $\hat{U}^{(\nu+3)}_{i_0 \dots i_k} \cap \hat{V}^{(\nu+3)}_{\iota_0 \dots \iota_\kappa} \subset \subset (R^{(\nu+2)}_{i_0 \dots i_k, \iota_0 \dots \iota_\kappa})_i$ for each $i \in \{ i_0, \dots, i_\kappa \}$. This inclusion shows that we get a well defined element $\hat{\eta} \in C^{k,\kappa-1}_{\nu+3}$ by restricting the elements $\hat{\eta}_{i_0,\dots i_k}$ to $C^{k,\kappa-1}_{\nu+3}$. We find that $\hat{\xi} \mid C^{k,\kappa}_{\nu+3} = \partial\hat{\eta}$ now. The norm inequalities are not obvious, but recalling how $\hat{\eta}$ is constructed here it is seen that we can apply Theorem I to obtain the required estimate.

Smoothing Theorem. There exists a constant K such that: If $\hat{\xi} \in \in Z^l(\hat{\mathfrak{B}}(\rho), \mathbf{F})$ with $\| \hat{\xi} \|_\rho < \infty$ then we can find $\hat{\xi}^* \in Z^l(\hat{\mathfrak{U}}(\rho), \mathbf{F})$ and $\hat{\eta} \in C^{l-1}(\hat{\mathfrak{B}}'(\rho). \mathbf{F})$ for which $\hat{\xi}^* \mid \hat{\mathfrak{B}}'(\rho) = \hat{\xi} \mid \hat{\mathfrak{B}}'(\rho) + \delta\hat{\eta}$ and $\| \hat{\xi}^* \|_\rho$ and $\| \hat{\eta} \|_\rho \leqslant K \| \hat{\xi} \|_\rho$. Here $\rho \leqslant \rho_2 < \rho_1$ and K only depends on ρ_2.

Proof. Before we can use the double complex $\{ C^{k,\kappa}_\nu \}$ we must introduce two " ε-maps ". To define the ε_1-map, let $Z^{k,\kappa}_\nu \subset C^{k,\kappa}_\nu$ consist of all $\hat{\xi} \in C^{k,\kappa}_\nu$ such that $\delta\hat{\xi} = \partial\hat{\xi} = 0$. Now we shall define the ε_1-map : ε_1 : $Z^l(\hat{\mathfrak{B}}, \mathbf{F}) \to Z^{0,l}_0$. A section belonging to an element of $C^{0,l}_0$ is defined on some set $\hat{U}^{(0)}_{i_0} \cap \hat{V}^{(0)}_{\iota_0,\dots \iota_l} \subset \hat{V}_{\iota_0 \dots \iota_l}$ where sections of elements of $Z^l(\hat{\mathfrak{B}}, \mathbf{F})$ are defined. Hence we get a natural restriction map ε_1 which also maps cocycles into cocycles. It is easy to verify that $\| \varepsilon_1(\hat{\xi}) \|_\rho \leqslant K \| \hat{\xi} \|_\rho$. Theorem I can be used because $(U^{(1)}_i \cap V^{(1)}_{\iota_0 \dots \iota_l})_i \subset (V^{(0)}_{\iota_0 \dots \iota_l})_\iota$ for every i and every $\iota \in \{ \iota_0, \dots \iota_l \}$. Recall that the norm in $Z^l(\hat{\mathfrak{B}}, \mathbf{F})$ is defined with respect to

$\hat{\mathfrak{B}}_0$ here. The "ε_2-map": we shall construct a map ε_2: $\hat{Z}^{l,0}_{3l} \to Z^l(\hat{\mathfrak{U}}, \mathbf{F})$. Let $\hat{\xi} = \{\hat{\xi}_{i_0,\ldots i_l,\iota_0}\} \in \hat{Z}^{l,0}_{3l}$. Here $\hat{\xi}_{i_0,\ldots i_l,\iota_0}$ is defined on $\hat{R}^{(3l)}_{i_0\ldots i_l,\iota_0}$. Because $\partial\hat{\xi} = 0$ we see that the elements $\hat{\xi}_{i_0\ldots i_l,\iota_0}$ are independent of ι_0. Now $\bigcup_{\iota=1}^{\iota^*} \hat{V}^{(3l)}_\iota$ covers $X(\rho_1)$. If we put $\varepsilon_2(\hat{\xi})_{i_0\ldots i_l} = \hat{\xi}_{i_0\ldots i_l,\iota_0}$ in $\hat{U}^{(3l)}_{i_0\ldots i_l} \cap \hat{V}^{(3l)}_{\iota_0}$ then we see that $\varepsilon_2(\hat{\xi})_{i_0\ldots i_l}$ is a well defined section on $\hat{U}^{(3l)}_{i_0\ldots i_l}$. In this way we obtain $\varepsilon_2(\hat{\xi}) \in Z^l(\hat{\mathfrak{U}}, \mathbf{F})$. Here $\varepsilon_2(\hat{\xi})$ is a cocycle because $\delta\hat{\xi} = 0$. Now we prove that $\| \varepsilon_2(\hat{\xi}) \|_\rho \leqslant K \| \hat{\xi} \|_\rho$.

Verification. A computation of $\| \varepsilon_2(\hat{\xi}) \|_\rho$ involves the following: $\varepsilon_2(\hat{\xi}) = \{\xi^{(2)}_{i_0}\ldots_{i_l}\}$. Look at some $\xi^{(2)}_{i_0\ldots i_l}$ in the chart $\mathscr{W}_i$ with $i \in \{i_0, \ldots, i_l\}$. We write $\hat{\xi}^{(2)}_{i_0\ldots i_l} = \sum a_\nu (t/\rho)^\nu$ over $(U^*_{i_0}\ldots_{i_l})_i$ and compute $\sup_\nu | a_\nu (U^*_{i_0}\ldots_{i_l}) |$. A computation of $\| \hat{\xi} \|_\rho$ involves the following: Look at $\hat{\xi}_{i_0\ldots i_l}$ over $(U^*_{i_0}\ldots_{i_l} \cap \cap V^*_\iota)_i$ in a chart W_i. Here ι is fixed. We write $\hat{\xi}_{i_0\ldots i_l,\iota} = \sum a_\nu^{(\iota)} (t/\rho)^\nu$ and compute $\sup_\nu | a_\nu^{(\iota)} (U^*_{i_0},\ldots_{i_l} \cap V^*_\iota) |$. Now $\bigcup_1^{\iota^*} V^*_\iota$ covers X_0. Hence we would have $\sup_{\nu,\iota} | a_\nu^{(\iota)}(U^*_{i_0}\ldots_{i_l} \cap V^*_\iota) | = \sup_\nu | a_\nu (U^*_{i_0}\ldots_{i_l}) |$ if $a_\nu = a_\nu^{(\iota)}$ in $U^*_{i_0}\ldots_{i_l} \cap \cap V^*_\iota$. But this is obvious since $\xi^{(2)}_{i_0}\ldots_{i_l} = \hat{\xi}_{i_0\ldots i_l,\iota}$ in $(U^*_{i_0}\ldots_{i_l} \cap V^*_\iota)_i$. Hence we have $\| \varepsilon_2(\hat{\xi}) \|_\rho \leqslant \| \hat{\xi} \|_\rho$.

Now we are ready to start the proof of the smoothing theorem. We let K denote a constant, which may be different at different occurences. We also introduce a double complex $\{\tilde{C}^{k,\kappa}_\nu\}$ using $(\mathfrak{B}, \mathfrak{B})$, i.e. it is defined just as the previous double complex was, using $\mathfrak{B}$-sets instead of $\mathfrak{U}$-sets. We shall inductively construct the following elements:

$$\hat{\xi}_\nu = \{\hat{\xi}_{i_0\ldots i_\nu,\iota_0\ldots\iota_{l-\nu}}\} \in \hat{Z}^{\nu,l-\nu}_{3\nu}$$

$$\tilde{\xi}_\nu = \{\tilde{\xi}_{i_0\ldots i_\nu,\iota_0\ldots\iota_{l-\nu}}\} \in \tilde{Z}^{\nu,l-\nu}_{3\nu};\ \nu = 0,\ldots,l$$

$$\hat{\eta}_\nu = \{\hat{\eta}_{i_0\ldots i_{\nu-1},\iota_0\ldots\iota_{l-\nu}}\} \in \hat{C}^{\nu-1,l-\nu}_{3\nu}$$

$$\tilde{\eta}_\nu = \{\tilde{\eta}_{i_0\ldots i_{\nu-1},\iota_0\ldots\iota_{l-\nu}}\} \in \tilde{C}^{\nu-1,\nu-1}_{3\nu};\quad \nu = 1,\ldots,l$$

$$\tilde{\gamma}_\nu = \{ \tilde{\gamma}_{i_0 \ldots i_{\nu-1}, \iota_0 \ldots \iota_{l-\nu-1}} \} \in \tilde{C}_{3\nu-3}^{\nu-1, l-\nu-1}; \quad \nu = 1, \ldots, (l-1)$$

and $\tilde{\gamma}_l = \{ \tilde{\gamma}_{i_0 \ldots i_{l-1}} \} \in C^{l-1} (\mathfrak{V}_{3l})$.

The construction: $\hat{\xi} \in Z^l (\hat{\mathfrak{V}} (\rho), \mathbf{F})$ is given. The whole construction is done using ρ instead of ρ_1 and we omit ρ to simplify the notation. We put $\varepsilon_1 (\hat{\xi}) = \hat{\xi}_0 \in Z^{0,l}_0$. Now we apply the Smoothing Lemma and get $\hat{\eta}_1$ such that $\partial \hat{\eta}_1 = \hat{\xi}_0$ with $\| \hat{\eta}_1 \|_\rho \leqslant K \| \hat{\xi}_0 \|_\rho \leqslant K \| \hat{\xi} \|_\rho$. Put $\hat{\xi}_1 = \delta \hat{\eta}_1$. Obviously $\| \hat{\xi}_1 \|_\rho \leqslant K \| \hat{\eta}_1 \|_\rho$. Inductively we find $\delta \hat{\eta}_\nu = \hat{\xi}_{\nu-1}$ and we put $\hat{\xi}_\nu = \delta \hat{\eta}_\nu$ where η_ν are found from the Smoothing Lemma. Finally we get $\hat{\xi}_l$ and we have $\| \hat{\xi}_l \|_\rho \leqslant K \| \hat{\xi} \|_\rho$. Now we define $\tilde{\xi}_\nu$ and $\tilde{\eta}_\nu$ as follows. Put $\tilde{\xi}_0 = \hat{\xi}_0$ where $\tilde{\xi}_0 \in \tilde{Z}^{0,l}_0$ is obtained by natural restriction of $\hat{\xi}_0$. Put $\tilde{\eta}_\nu = (-1)^\nu \{ \hat{\xi}_{i_0 \ldots i_{\nu-1}, \iota_0 \ldots \iota_{l-\nu}} \}$ which is well defined with respect to $(\mathfrak{V}_{3\nu}, \mathfrak{V}_{3\nu})$ by taking natural restrictions. Put $\tilde{\xi}_\nu = \delta \tilde{\eta}_\nu$ for $\nu = 1, \ldots, l$. A computation shows that $\tilde{\xi}_{\nu-1} = \partial \tilde{\eta}_\nu$ when $\nu = 1, \ldots, l$. Notice that this is trivial when $\nu = 1$. In the following discussion each $\hat{\eta}_\nu$ is restricted to $(\mathfrak{V}_{3\nu}, \mathfrak{V}_{3\nu})$. We have $\partial (\tilde{\eta}_1 - \hat{\eta}_1) = 0$. Hence we find $\tilde{\eta}_1 - \hat{\eta}_1 = \partial \tilde{\gamma}_1$ by the Smoothing Lemma. Now we define $\tilde{\gamma}_\nu$ such that $\partial \tilde{\gamma}_\nu = \tilde{\eta}_\nu - \hat{\eta}_\nu - \delta \tilde{\gamma}_{\nu-1}$ inductively. This is possible because $\partial (\tilde{\eta}_\nu - \hat{\eta}_\nu - \delta \tilde{\gamma}_{\nu-1}) = 0$, for we have $\partial (\tilde{\eta}_\nu - \hat{\eta}_\nu - \delta \tilde{\gamma}_{\nu-1}) = \tilde{\xi}_{\nu-1} - \hat{\xi}_{\nu-1} - \delta \partial \tilde{\gamma}_{\nu-1} = \delta \tilde{\eta}_{\nu-1} - \delta \hat{\eta}_{\nu-1} - \delta (\tilde{\eta}_{\nu-1} - \hat{\eta}_{\nu-1}) = 0$. We get finally $\tilde{\gamma}_{l-1} \in \tilde{C}^{l-2,0}_{3l}$ and then $\delta \tilde{\gamma}_{l-1} \in \tilde{C}^{l-1,0}_{3l}$. We have $\partial (\tilde{\eta}_l - \hat{\eta}_l - \delta \tilde{\gamma}_{l-1}) = 0$. Therefore we can put $\tilde{\gamma}_l = \varepsilon_2 (\tilde{\eta}_l - \hat{\eta}_l - \delta \tilde{\gamma}_{l-1})$. It follows that $\tilde{\gamma}_l \in C^{l-1} (\mathfrak{V}_{3l})$ and $\delta \tilde{\gamma}_l = \varepsilon_2 (\tilde{\xi}_l - \hat{\xi}_l)$. We have $\varepsilon_2 (\tilde{\xi}_l) = - \hat{\xi} \mid \mathfrak{V}'$ and for $\varepsilon_2 (\hat{\xi}_l) = - \hat{\xi}^*$ and $\hat{\eta} = \tilde{\gamma}_l$ the required equation $\hat{\xi}^* = \hat{\xi} + \delta \hat{\eta}$. The estimates follow immediately from the construction and the Smoothing Lemma.

Approximation

We use positive n-tuples $\rho, \ldots$ with $\rho \leqslant \rho_2 < \rho_3 < \rho_4 < \rho_1$ and $\rho = \gamma'' \rho_1, \rho_2 = \gamma \rho_1, \rho_3 = \gamma' \rho_1, \rho_4 = \gamma''' \rho_1$. The n-tuple ρ_1 is defined as in the smoothing theorem.

Definition: $H^l_* = \{ \xi \in H^l(X_0, \underline{F|X_0})$ such that there exists $U = U(0)$ in E^n with $\hat{\xi} \in H^l(\psi^{-1}(U), \mathbf{F})$ and $\hat{\xi} \mid X_0 = \xi \}$. Serre's theorem gives $\dim_{\mathbf{C}} H^l_* \leqslant \dim_{\mathbf{C}} H^l(X_0, \underline{F|X_0}) < \infty$. In the following discussion we are given $\hat{\mathfrak{b}}_1, \ldots, \hat{\mathfrak{b}}_r$ in $Z^l(\hat{\mathfrak{U}}'(\rho_4), \mathbf{F})$ such that $\hat{\mathfrak{b}}_1 \mid X_0, \ldots \hat{\mathfrak{b}}_r \mid X_0$ constitute a base of the complex vector space H^l_*. For this to be possible, ρ_4 has to be chosen small enough. Here $\hat{\mathfrak{U}}'$ is a Stein covering of $X(\rho_1)$ and defined as in the smoothing theorem. We also assume that we are given a sequence of measure coverings as there. Further we construct the sequence so that there are still sufficiently many measure coverings in between $\mathfrak{B}$ and $\mathfrak{U}$. These are denoted by $\mathfrak{U}^*_\nu$. We have $\mathfrak{U} \gg \mathfrak{U}^*_1 \gg \mathfrak{U}^*_2 \gg \ldots \gg \mathfrak{B}$. The n-tupel ρ_3 is also fixed from now on and K always denotes (possibly different) constants.

Approximation Lemma: Let $\varepsilon > 0$. Then we can find ρ_2 such that: If $\rho \leqslant \rho_2$ and $\hat{\xi} \in Z^l(\hat{\mathfrak{U}}(\rho), \mathbf{F})$ with $\| \hat{\xi} \|_\rho < \infty$ (the norm is taken with respect to $\hat{\mathfrak{U}}^*_1(\rho)$), then there exist $a_1, \ldots a_r \in I(E^n(\rho))$ and $\hat{\eta} \in C^{l-1}(\mathfrak{B}(\rho), \mathbf{F})$ such that $\tilde{\xi} = \hat{\xi} - \sum_1^r a_i \hat{\mathfrak{b}}_i - \delta\hat{\eta}$ on $\mathfrak{B}(\rho)$. Here $\tilde{\xi} \in Z^l(\mathfrak{B}(\rho), \mathbf{F})$ and $\| \tilde{\xi} \|_\rho \leqslant \varepsilon \| \hat{\xi} \|_\rho$ and $\| a_\nu \|_\rho, \| \hat{\eta} \|_\rho \leqslant K \| \hat{\xi} \|_\rho$. K is a fixed constant.

Proof. We shall first prove some results which are needed later on. Let $S \in \Gamma(\hat{U}_{\iota_0 \ldots \iota_l}(\rho), \mathbf{F})$. Choose $\iota \in \{ \iota_0, \ldots, \iota_l \}$. Now $(U^{(1)*}_{\iota_0 \ldots \iota_l})_\iota \subset \hat{U}_{\iota_0 \ldots \iota_l}$ because $\mathfrak{U}^*_1 \ll \mathfrak{U}$. The operations are always defined with respect to ρ_1. We can now restrict S to $(U^{(1)*}_{\iota_0 \ldots \iota_l})_\iota(\rho)$. In the chart $\mathscr{W}_\iota$ we can write $S = \sum a_\nu (t/\rho)^\nu$. Here $a_\nu \in qI(U^{(1)*}_{\iota_0 \ldots \iota_l})$. Now the a_ν are extended constantly and we get elements $\hat{a}_\nu \in \Gamma((U^{(1)*}_{\iota_0 \ldots \iota_l})_\iota, \mathbf{F})$. Let us put $S_\nu = \hat{a}_\nu \mid \hat{U}^{(2)*}_{\iota_0 \ldots \iota_l}$. We claim that $\| S_\nu \|_{\rho_1} \leqslant K \| S \|_\rho$. For obviously $\| S \|_\rho \geqslant | a_\nu (U^{(1)*}_{\iota_0 \ldots \iota_l}) |$ and we can use the Theorem I to prove that $\| S_\nu \|_{\rho 1} \leqslant K \| \hat{a}_\nu \mid (U^{(1)*}_{\iota_0 \ldots \iota_l})_\iota (\rho_1) \|_\iota = K \mid a_\nu (U^{(1)*}_{\iota_0 \ldots \iota_l}) \mid \leqslant K \| S \|_\rho$. Q.E.D.

Let S_ν' be defined using some other $\iota' \in \{ \iota_0, \ldots \iota_l \}$. Then $S_\nu - S_\nu' \in \Gamma (\hat{U}^{(2)*}_{\iota_0 \ldots \iota_l}, \mathbf{F})$. We claim that $\| S_\nu - S_\nu' \|_{\rho_4} \leqslant K \gamma''' \| S \|_\rho$.

Proof. Define $\alpha_s = \sum_{|\lambda| = s}^{\infty} a_\lambda (t/\rho)^\lambda$ and $\beta_s = \sum_{|\lambda|=0}^{s-1} a_\lambda (t/\rho)^\lambda$ over $(U^{(1)*}_{\iota_0 \ldots \iota_l})_\iota (\rho)$. We do the same for ι' respectively and obtain α_s' and β_s' over $(U^{(1)*}_{\iota_0 \ldots \iota_l})_{\iota'} (\rho)$. For the restrictions to $\hat{U}^{(2)*}_{\iota_0 \ldots \iota_l}$ we see that $\alpha_s - \alpha_s' = - (\beta_s - \beta_s')$. Hence we get $\| \alpha_s - \alpha_s' \|_{\rho_4} \leqslant K (\gamma''')^s \| \alpha_s - \alpha_s' \|_{\rho_1} = K (\gamma''')^s \| \beta_s - \beta_s \|_{\rho_1} \leqslant K (\gamma''')^s \| \beta_s \|_{\rho_1} + K (\gamma''')^s \| \beta_s' \|_{\rho_1} \leqslant K (\gamma''')^s [\| \beta_s \|_{\rho_1}^* + \| \beta_s' \|_{\rho_1}^*] \leqslant K (\gamma''')^s (\gamma'')^{1-s} \| S \|_\rho$. Here the norms are defined with respect to $U^{(3)*}_{\iota_0 \ldots \iota_l}$ except $\| \, \|^*$ and $\| S \|_\rho$ which are defined with respect to $U^{(1)*}_{\iota_0 \ldots \iota_l}$. Now we look at the difference $(S_\nu - S_\nu') t^\nu / \rho^\nu$ on $(U^{(3)*}_{\iota_0 \ldots \iota_l})_\mu$ with $|\nu| = s$, $\mu \in \{ \iota_0, \ldots \iota_l \}$, and the power series development with respect to W_μ. There is one term of order s which is equal to the corresponding term of $\alpha_s - \alpha_s'$. Therefore its norm is $\leqslant K (\gamma''')^s . (\gamma'')^{1-s} \| S \|_\rho$. Moreover we have $\| S_\nu (t/\rho)^\nu - S_\nu' (t/\rho)^\nu \|_{\rho_1} \leqslant (\gamma'')^{-s} \cdot K \| S \|_\rho$ where the first norm is defined with respect to $U^{(3)*}_{\iota_0 \ldots \iota_l}$. For the sum $\sum$ of terms of higher order than s in the power series of $(S_\nu - S_\nu') t^\nu / \rho^\nu$ we therefore get: $\| \sum \|_{\rho_4} \leqslant (\gamma''')^{s+1} (\gamma'')^{-s} \cdot K \| S \|_\rho$. Hence we get $\| (S_\nu - S_\nu') \|_{\rho_4} \leqslant \gamma''' \cdot K \| S \|_\rho$. This proves our statement. We see that K is independent of ρ_4 and S. The number γ''' depends on ρ_4 only, so $\gamma''' \cdot K$ gets very small if we make ρ_4 very small.

Let $\hat{\xi} \in Z^l (\hat{\mathfrak{U}} (\rho), \mathbf{F})$ with $\hat{\xi} = \{ \hat{\xi}_{\iota_0 \ldots \iota_l} \}$. Choose $\iota = \iota (\iota_0, \ldots, \iota_l)$ as a function of the unordered $(l+1)$-tuple. We now fix $\iota_0, \ldots, \iota_l$ and write $S = \hat{\xi}_{\iota_0 \ldots \iota_l}$. We apply to S the method described above and obtain $\hat{\xi}^{(\nu)}_{\iota_0 \ldots \iota_l} = S_\nu$. We do this now for every $\iota_0, \ldots, \iota_l$ and consider $\hat{\xi}_{(\nu)} = \{ \hat{\xi}^{(\nu)}_{\iota_0 \ldots \iota_l} \}$ as an element of $C^l (\hat{\mathfrak{U}}_2^* (\rho_4), \mathbf{F})$. Of course $\hat{\xi}_{(\nu)}$ depends on the choice of $\iota = \iota (\iota_0 \ldots \iota_l)$ here. Now we see that $\| \hat{\xi}_{(\nu)} \|_{\rho_4} \leqslant \| \hat{\xi}_{(\nu)} \|_{\rho_1} \leqslant K \| \hat{\xi} \|_\rho$. We also wish to estimate $\delta \hat{\xi}_{(\nu)}$. Because $\hat{\xi} \in Z^l (\hat{\mathfrak{U}} (\rho), \mathbf{F})$ we can use the preliminary result on ι and ι' to obtain $\| \delta \hat{\xi}_{(\nu)} \|_{\rho_4} \leqslant K \gamma''' \| \hat{\xi} \|_\rho$.

We shall also need another result:

Induction Lemma: There exists $\hat{\eta}_\nu \in C^l(\mathfrak{U}_4^*(\rho_3), \mathbf{F})$ such that $\delta\hat{\eta}_\nu = \delta\hat{\xi}_{(\nu)}$ on $\hat{\mathfrak{U}}_4^*(\rho_3)$ and $\| \hat{\eta}_\nu \|_{\rho_3} \leqslant K \| \delta\hat{\xi}_{(\nu)} \|_{\rho_4}$.

Proof. The proof uses the assumption that $\psi_{(l+1)}(\mathbf{F})$ is coherent. Because the coherence of direct images is proved by downward induction on l, this assumption can be made. Moreover it is assumed that the main theorem is proved for dimension $l+1$ already. Let us now put $\alpha = \delta\hat{\xi}_{(\nu)} \in B^{l+1}(\hat{\mathfrak{U}}_2^*(\rho_4), \mathbf{F})$ and $\hat{\eta}_\nu = \beta \in C^l(\hat{\mathfrak{U}}_4^*(\rho_3), \mathbf{F})$. We have to prove the existence of β. We may assume that ρ_4 is so small that the main theorem is valid for $\rho \leqslant \rho_4$ in the case of dimension $l+1$. So there are cocycles $\omega_1, \ldots, \omega_r \in Z^{l+1}(\hat{\mathfrak{U}}(\rho_4), \mathbf{F})$ such that $\alpha = \sum C_\lambda \omega_\lambda + \delta\eta$, where $C_\lambda \in I(E^n(\rho_4))$ and $\eta \in C^l(\hat{\mathfrak{U}}_4^*(\rho_4), \mathbf{F})$. We have to assume that between $\hat{\mathfrak{U}}_4^*$ and $\mathfrak{U}_2^*$ there are very many measure coverings. The cross-sections $\psi_{(l+1)}(\omega_\lambda)$ give a homomorphism $r\mathcal{O} \to \psi_{(l+1)}(\mathbf{F})$ over $E^n(\rho_4)$. Because $\psi_{(l+1)}(\mathbf{F})$ is coherent the kernel $\mathcal{N}$ is coherent again. Over $E^n(\rho')$ with $\rho_3 < \rho' < \rho_4$ we find an epimorphism $p\mathcal{O} \to \mathcal{N}$. Denote by $n_1, \ldots, n_p$ the images of the unit cross-sections in $p\mathcal{O}$. Write $n_\lambda = (e_{\lambda 1}, \ldots, e_{\lambda r})$ as an r-tupel of holomorphic functions. The image of n_λ in $\Gamma(E^n(\rho'), \psi_{(l+1)}(\mathbf{F}))$ is $\psi_{(l+1)}(\sum_{\mu=1}^{r} e_{\lambda\mu}\omega_\mu)$ and zero. We may choose ρ_2 and then ρ_3 and ρ' very small. Then it follows that $\hat{n}_\lambda = \sum e_{\lambda\mu}\omega_\mu$ is a coboundary. If $\rho_3 < \rho'' < \rho'$ there are cochains $\eta_\lambda \in C^l(\hat{\mathfrak{U}}_4^*(\rho''), \mathbf{F})$ such that $\delta\eta_\lambda = \hat{n}_\lambda$. Now $(C_1, \ldots, C_r) \in \Gamma(E^n(\rho_4), \mathcal{N})$. By the methods of sheaf theory we can lift this cross-section to $p\mathcal{O}$. Using a "Banach open mapping theorem" we see that the map $\Gamma(E^n(\rho'), p\mathcal{O}) \to \Gamma(E^n(\rho'), \mathcal{N})$ is open. This means here that we can find holomorphic functions a_λ over $E^n(\rho_3)$ such that $C_\mu = \sum a_\lambda e_{\lambda\mu}$ and $\| a_\lambda \|_{\rho_3} \leqslant K \max_\mu \| C_\mu \|_{\rho'} \leqslant K \max_\mu \| C_\mu \|_{\rho_4}$. We get $\sum C_\mu \omega_\mu = \sum a_\lambda e_{\lambda\mu}\omega_\mu = \sum a_\lambda \hat{n}_\lambda = \delta(\sum a_\lambda \eta_\lambda)$. This leads to $\alpha \mid C^{l+1}(\hat{\mathfrak{U}}_4^*(\rho_3)) = \delta(\eta + \sum a_\lambda \eta_\lambda)$. The estimates required obviously hold. Q.E.D.

Let us now put $\hat{\xi}_{(\nu)}^* = \hat{\xi}_{(\nu)} - \hat{\eta}_\nu \in Z^l(\hat{\mathfrak{U}}_4(\rho_3), \mathbf{F})$. We can write $\hat{\xi}_{(\nu)}^* \mid X_0 = \sum a_{\nu\lambda} \hat{\mathfrak{b}}_\lambda \mid X_0 + \delta\gamma_\nu$ over $\mathfrak{U}_6^*$. Here $a_{\nu\lambda}$ are complex numbers and $\gamma_\nu \in C^{l-1}(\mathfrak{U}_6^*, F \mid X_0)$. Cartan's theorem and the result after that give the estimates $| a_{\nu\lambda} | \leqslant K \| \hat{\xi}_{(\nu)}^* \|_{\rho_3} \leqslant K \| \hat{\xi} \|_\rho$ and $\| \hat{\gamma}_\nu \|_{\rho_3} \leqslant K \| \hat{\xi}_{(\nu)}^* \|_{\rho_3} \leqslant K \| \hat{\xi} \|_\rho$. Here $\hat{\gamma}_\nu \in C^{l-1}(\hat{\mathfrak{U}}_7^*(\rho_3), \mathbf{F})$ has been obtained by a constant extension of γ_ν. Let us now put $\hat{\xi}^{(1)}_{(\nu)} = \hat{\xi}^*_{(\nu)} - \sum a_{\nu\lambda} \hat{\mathfrak{b}}_\lambda - \delta\hat{\gamma}_\nu$. Here $\hat{\xi}^{(1)}_{(\nu)} \in$ $\in C^l(\hat{\mathfrak{U}}^*_7(\rho_3), \mathbf{F})$. Using the previous estimates and the fact that the $\hat{\mathfrak{b}}_\lambda$ are finite we find that $\| \hat{\xi}^{(1)}_{(\nu)} \|_{\rho_3} \leqslant K \| \hat{\xi}_{(\nu)} \|_{\rho_4} \leqslant K \| \hat{\xi} \|_\rho$.

Now we also have $\hat{\xi}^{(1)}_{(\nu)} \mid X_0 = 0$. It follows that

$$\| \hat{\xi}^{(1)}_{(\nu)} \|_\rho \leqslant \gamma/\gamma' \, \| \hat{\xi}^{(1)}_{(\nu)} \|_{\rho_3} \leqslant \gamma/\gamma' \cdot K \, \| \hat{\xi} \|_\rho \,.$$

Finally we put in $\hat{\mathfrak{U}}^*_9(\rho)$:

$$\hat{\xi}^{(1)} = \Sigma \hat{\xi}^{(1)}_{(\nu)} (t/\rho)^\nu =$$

$$= \Sigma \hat{\xi}_{(\nu)} (t/\rho)^\nu - \Sigma \hat{\eta}_\nu (t/\rho)^\nu - \Sigma a_{\nu\lambda} (t/\rho)^\nu \hat{\mathfrak{b}}_\lambda - \delta(\Sigma \hat{\gamma}_\nu (t/\rho)^\nu)$$

$$= \hat{\xi} - \hat{\eta} - \Sigma a_\lambda \hat{\mathfrak{b}}_\lambda - \delta\hat{\gamma} \,.$$

Using the fact that the sum of the absolute values of the coefficients in the power series expansion of $\hat{\xi}^{(1)}_{(\nu)}$ by (t/ρ) is smaller than $\gamma/\gamma' \cdot K \| \hat{\xi} \|_\rho$ and that with respect to $\hat{\eta}_\nu$ is smaller than $\gamma''' \cdot K \| \hat{\xi} \|_\rho$ we find: $\| \hat{\xi}^{(1)} \|_\rho \leqslant$ $\leqslant \gamma/\gamma' \cdot K \| \hat{\xi} \|_\rho$ and $\| \hat{\eta} \|_\rho \leqslant \gamma''' \cdot K \| \hat{\xi} \|_\rho$ and $\| a_\lambda \|_\rho \leqslant K \| \hat{\xi} \|_\rho$. We take the restriction to $\hat{\mathfrak{B}}(\rho)$ and now $\tilde{\xi} = \hat{\xi}^{(1)} - \hat{\eta} \in Z^l(\hat{\mathfrak{B}}(\rho), \mathbf{F})$ is the desired element. Of course we have to choose ρ_4 and then ρ_2 small enough, for example let $\gamma''' < \varepsilon/2K$ and $\gamma \leqslant \varepsilon\gamma'/2K$.

Main Theorem

There exists ρ_2 and a constant K such that if $\rho \leqslant \rho_2$ and $\hat{\xi} \in Z^l(\hat{\mathfrak{U}}(\rho), \mathbf{F})$ with $\| \hat{\xi} \|_\rho < \infty$ then we can find $a_1, \ldots, a_r \in I(E^n(\rho))$ and $\hat{\eta} \in$ $C^{l-1}(\hat{\mathfrak{B}}(\rho), \mathbf{F})$ such that $\hat{\xi} = \sum a_\lambda \hat{\mathfrak{b}}_\lambda + \delta\hat{\eta}$ on $\hat{\mathfrak{B}}(\rho)$ with $\| \hat{\eta} \|_\rho$ and $\| a_\nu \|_\rho \leqslant$ $\leqslant K \| \hat{\xi} \|_\rho$.

Proof. We have one constant K from the smoothing theorem. Now we find ρ_2 with an ε in the Approximation Lemma such that $\varepsilon \cdot K < 1/2$. We shall use this ρ_2 and prove the theorem here. We are given $\hat{\xi}_0 = \hat{\xi} \in$ $Z^l(\hat{\mathfrak{U}}(\rho), \mathbf{F})$ with $\| \hat{\xi} \|_\rho < \infty$. The Approximation Lemma gives $\tilde{\xi}_1 =$

$= \hat{\xi} - \sum \hat{a}_{1\lambda} \mathfrak{b}_\lambda - \delta\hat{\gamma}_1$ on $\hat{\mathfrak{B}}(\rho)$. Here $\hat{\gamma}_1 \in C^{l-1}(\hat{\mathfrak{B}}(\rho), \mathbf{F})$ and $\| \tilde{\xi}_1 \|_\rho \leqslant \leqslant \varepsilon \| \hat{\xi} \|_\rho$. Now $\tilde{\xi}_1 \in Z^l(\mathfrak{B}(\rho), \mathbf{F})$. The Smoothing Theorem gives $\hat{\xi}_1 \in \in Z^l(\hat{\mathfrak{U}}(\rho), \mathbf{F})$ and $\hat{\eta}_1 \in C^{l-1}(\hat{\mathfrak{B}}(\rho), \mathbf{F})$ such that $\hat{\xi}_1 = \tilde{\xi}_1 + \delta\hat{\eta}_1$ on $\hat{\mathfrak{B}}'(\rho)$. Here $\| \hat{\eta}_1 \|_\rho$ and $\| \hat{\xi}_1 \|_\rho \leqslant K \| \tilde{\xi}_1 \|_\rho < \frac{1}{2} \| \hat{\xi} \|_\rho$. Now we use $\hat{\xi}_1$ instead of $\hat{\xi}_0$ as above and get: $\hat{\xi}_2 = \hat{\xi}_1 + \delta\hat{\eta}_2 - \sum a_{2\lambda} \hat{\mathfrak{b}}_\lambda - \delta\hat{\gamma}_2$. Here $\| \hat{\xi}_2 \|_\rho$ and $\| \hat{\eta}_2 \| < \frac{1}{2} \| \hat{\xi}_1 \|_\rho < (\frac{1}{2})^2 \| \hat{\xi} \|_\rho$ and $\| a_{2\lambda} \|_\rho$ and $\| \hat{\gamma}_2 \|_\rho \leqslant \frac{K}{2} \| \hat{\xi} \|_\rho$. Inductively we get: $\hat{\xi}_n = \hat{\xi}_{n-1} - \sum a_{n\lambda} \hat{\mathfrak{b}}_\lambda - \delta\hat{\gamma}_n + \delta\hat{\eta}_n$. Here $\| \hat{\xi}_n \|_\rho < < 2^{-n} \| \hat{\xi} \|_\rho$, $\| \hat{\eta}_n \|_\rho \leqslant 2^{-n} \| \hat{\xi} \|_\rho$ and $\| a_{n\lambda} \|_\rho$ and $\| \hat{\gamma}_n \|_\rho \leqslant 2^{-n+1} \cdot K \| \hat{\xi} \|_\rho$ for $n = 1, 2, 3, \ldots$ A summation is now possible. We get $0 = \hat{\xi} - \sum_{n,\lambda} a_{n\lambda} \hat{\mathfrak{b}}_\lambda - - \sum \delta\hat{\gamma}_n + \sum \delta\hat{\eta}_n$. We put $a_\lambda = \sum_n a_{n\lambda}$, $\hat{\eta} = \sum(-\hat{\gamma}_n + \hat{\eta}_n)$ and the theorem follows.

For the proof of the coherence the Main Theorem is needed in a weaker and simpler form.

Main Theorem (*): There exists a positive n-tupel $\rho_2 \leqslant \rho_0$ and cross-sections $S_1, \ldots, S_r \in \Gamma(E^n(\rho_2), \psi_{(l)}(\mathbf{F}))$ such that any $S = \psi_{(l)}(\underline{\hat{\xi}}') \in \Gamma(E^n(\rho'), \psi_{(l)}(\mathbf{F}))$ with $\underline{\hat{\xi}}' \in H^l(X(\rho'), \mathbf{F})$ can be written over $E^n(\rho)$ in the form $S = = \sum_1^n a_\lambda S_\lambda$ with $a_1, \ldots, a_r \in I(E^n(\rho))$. Here $\rho \leqslant \rho_2$ and $\rho < \rho' \leqslant \rho_0$.

Proof. Define $S_\lambda = \psi_{(l)}(\mathbf{b}_\lambda \mid X(\rho_2))$. The cross-section S can be written in the form $S = \psi_{(l)}(\hat{\xi}')$ with $\hat{\xi}' \in Z^l(\hat{\mathfrak{U}}'(\rho'), \mathbf{F})$. We put $\hat{\xi} = \hat{\xi}' \mid \hat{\mathfrak{U}}(\rho)$. Then $\| \hat{\xi} \|_\rho < \infty$ and we have the representation $\hat{\xi} = = \sum a_\lambda \hat{\mathfrak{b}}_\lambda + \delta\hat{\eta}$. For the cohomology classes we get $\underline{\hat{\xi}} = \sum a_\lambda \mathbf{b}_\lambda$ and for the images $S \mid E^n(\rho)$, this gives $S \mid E^n(\rho) = \psi_{(l)}(\hat{\xi}) = \sum a_\lambda S_\lambda$.

The immediate consequence of this form of the Main Theorem is that the stalk of $\psi_{(l)}(\mathbf{F})$ at the origin (and hence at every point of course) is finitely generated. However this is not yet the full coherence of $\psi_{(l)}(\mathbf{F})$. Nevertheless, the Main Theorem above contains all that is essential, and the rest of the proof is not difficult. We refer to [1, pp. 54-58], or to Knorr [2] for details.

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