3.2. Germs of analytic spaces.

Remarks. *1.* The same proof applies to the real case, and, more generally, to analytic algebras over a complete valuated field.

2. In the C^∞ case (over $\mathbf{R}$), it is known that the existence part of theorem 3.1.1. is true. Therefore steps 1 and 2 of the preceeding proof are applicable, but not step 3 (the lifting $\tilde{f}$ cannot be constructed a priori, so one has to *suppose* that such a lifting exists).

3.2. *Germs of analytic spaces.*

This concept will be introduced in terms of categories. As objects, we take triples $(X, \mathcal{O}_X, x)$ where $(X, \mathcal{O}_X)$ is an analytic space, and x a point of X ; as morphisms of $(X, \mathcal{O}_X, x)$ into $(Y, \mathcal{O}_Y, y)$ we take the germs at x of morphisms of $(X, \mathcal{O}_X)$ into $(Y, \mathcal{O}_Y)$, which map x into y. To simplify the notations, we write (X, x) for $(X, \mathcal{O}_X, x)$.

We shall prove some results on the correspondence between analytic rings and germs of analytic spaces.

Proposition 3.2.1. To any germ (X, x) of an analytic space is associated an analytic ring $\mathcal{O}_{X,x}$. Every analytic ring is obtained in this way. Every morphism $(X, x) \to (Y, y)$ of germs of analytic spaces induces a homomorphism $\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ of analytic rings. Conversely every homomorphism $B \to A$ of analytic rings is obtained from a morphism of corresponding germs of analytic spaces; the latter is unique.

Proof. If (X, x) is a germ of analytic spaces, $\mathcal{O}_{X,x}$ is an analytic ring by definition. Now let $A = \mathbf{C}\{x_1, ..., x_n\}/\mathrm{I}$ be an analytic ring. We choose generators $f_1, ..., f_p$ for I and take an open neighborhood U of 0 such that representatives of $f_1, ..., f_p$ which are analytic in U can be found. These generators then define a coherent sheaf $\mathcal{I}$ of ideals on U which defines an analytic subspace X of U with $\mathcal{O}_{X,0} = A$.

If $f: B \to A$ is a homomorphism of analytic rings, we shall construct a morphism $(X, 0) \to (Y, 0)$ of corresponding germs which induces F. We may suppose

$$A = \mathbf{C}\{x_1, ..., x_n\}/(f_1, ..., f_p), \qquad B = \mathbf{C}\{y_1, ..., y_m\}/(g_1, ..., g_q);$$

as we have seen in § 1, F can be lifted into a homomorphism $F^1 : \mathbf{C}\{y_1, ..., y_m\} \to \mathbf{C}\{x_1, ..., x_n\}$; we can choose 1) open sets $U \subset \mathbf{C}^n$, $V \subset \mathbf{C}^m$ with $0 \in U, 0 \in V$ 2) holomorphic functions $\bar{f}_1, ..., \bar{f}_p$ in U and $\bar{g}_1, ..., \bar{g}_q$ in V such that their germs at 0 are precisely the f_i's and the g_j's, and 3) an holomorphic mapping $\Phi : U \to V$, with $\Phi(0) = 0$ such that Φ^* induces F' at the origin.

Denote now by $\mathscr{I}$ (resp $\mathscr{J}$) the coherent sheaf of ideals generated in U (resp. V) by the $\bar{f}_i's$ | resp. the $\bar{g}'_js$). We have $\Phi^*(\mathscr{J})_0 \subset \mathscr{I}_0$, hence, since $\mathscr{J}$ is finitely generated by restricting U and V if necessary, we have $\Phi^*(\mathscr{J}) \subset \mathscr{I}$. Finally we take $X = \operatorname{supp} \mathcal{O}_U/\mathscr{I}$, $\mathcal{O}_X = \mathcal{O}_U/\mathscr{I} \mid_X$ and the same for Y; it is clear that Φ induces the required morphism $(Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$.

Finally, if two morphisms $\varphi, \psi : (X, 0) \to (Y, 0)$ induce the same homomorphism $\mathcal{O}_{Y,0} \to \mathcal{O}_{X,0}$, we have to prove that φ and ψ are equals. We may assume that Y is given by a local model $(Y, \mathcal{O}_V | \mathscr{J} | Y)$ for some coherent sheaf $\mathscr{J}$ of ideals on an open set $V \subset \mathbf{C}^m$; by composition with the injection $Y \to V$, we may restrict ourselves to the case where $Y = \mathbf{C}^m$; the morphisms φ and ψ are now given by sections $f, g \in \Gamma(X, \mathcal{O}_X{}^m)$, and the hypothesis means that the germs of f and g at 0 coincide; hence f and g coincide in a neighborhood of 0 in X, which proves the assertion.

3.3 *Finite morphisms*

Let $f : (X, 0) \to (Y, 0)$ be a morphism of germs of analytic spaces. Then f is called " finite " if the corresponding homomorphism $f^* : \mathcal{O}_{Y,0} \to \mathcal{O}_{X,0}$ makes $\mathcal{O}_{X,0}$ finite over $\mathcal{O}_{Y,0}$. According to the preparation theorem 3.1.3. in order that f be finite, it is necessary and sufficient that $\mathcal{O}_{X,0}/\mathfrak{M}(\mathcal{O}_{Y,0})\, \mathcal{O}_{X,0}$ be finite over $\mathbf{C}$; in geometrical terms, this means that the germ of space $f^{-1}(0)$ is finite over the point 0 (see § 1.3, example 4).

In the global case (complex or real), we give the following definition:

Definition 3.3.1. A morphism of *separated* analytic spaces $f = (f_0, f^1)$: $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is finite if the following properties hold:

1) f is proper (i.e. f_0 is proper).

2) For any point $x \in X$, the induced morphism of germs $f_x : (X, \mathcal{O}_X, x) \to (Y, \mathcal{O}_Y, f_0(x))$ is finite.

In the *complex* case, we have the following results :

Proposition 3.3.2. f is finite if and only if f is proper and, for any $b \in Y$, the *set* $f_0{}^{-1}(b)$ is finite.

This proposition is more or less equivalent to the " Nullstellensatz "; for the proof see e.g. Houzel [6] or Narasimhan [9]. In the real case, the part " if " of this proposition is not even true when Y is a point : for instance the subspace of $\mathbf{R}^2$ defined by $\mathscr{I}$ = (coherent sheaf of ideals generated by $x_1^2 + x_2^2$) has support 0 ; but $\mathbf{R}\{x_1, x_2\}/(x_1^2 + x_2^2)$ is not finite over $\mathbf{R}$.

Proposition 3.3.2. If $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a finite morphism, then the direct image $f_*(\mathcal{O}_X)$ is a coherent analytic sheaf of $\mathcal{O}_Y$-modules ; converse-