2. BOUNDEDNESS THEOREMS I

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **13 (1967)**

Heft 1: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **23.09.2024**

Nutzungsbedingungen

Die ETH-Bibliothek ist Anbieterin der digitalisierten Zeitschriften. Sie besitzt keine Urheberrechte an den Inhalten der Zeitschriften. Die Rechte liegen in der Regel bei den Herausgebern.
Die auf der Plattform e-periodica veröffentlichten Dokumente stehen für nicht-kommerzielle Zwecke in Lehre und Forschung sowie für die private Nutzung frei zur Verfügung. Einzelne Dateien oder Ausdrucke aus diesem Angebot können zusammen mit diesen Nutzungsbedingungen und den korrekten Herkunftsbezeichnungen weitergegeben werden.
Das Veröffentlichen von Bildern in Print- und Online-Publikationen ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Die systematische Speicherung von Teilen des elektronischen Angebots auf anderen Servern bedarf ebenfalls des schriftlichen Einverständnisses der Rechteinhaber.

Haftungsausschluss

Alle Angaben erfolgen ohne Gewähr für Vollständigkeit oder Richtigkeit. Es wird keine Haftung übernommen für Schäden durch die Verwendung von Informationen aus diesem Online-Angebot oder durch das Fehlen von Informationen. Dies gilt auch für Inhalte Dritter, die über dieses Angebot zugänglich sind.

Ein Dienst der *ETH-Bibliothek*
ETH Zürich, Rämistrasse 101, 8092 Zürich, Schweiz, www.library.ethz.ch

http://www.e-periodica.ch

Theorem (I). Suppose that assumptions A_1, A_2, A_3, and A_4 hold and in addition that $a(t) > 0$ and $a'(t) \geqq 0$ for $t \geqq T$. Then all solutions of (1.1) are bounded.

Corollary. In addition to the hypothesis of Theorem (I), suppose that assumption A_5 also holds and that $\lim_{t \to \infty} a(t) = k > 0$; then all solutions of (1.1) and their derivatives are bounded.

Theorem (II). Suppose that assumptions A_1, A_2, A_3 and A_4 hold and in addition that $a'(t) \leqq 0$ for $t \geqq T$. Then all solutions of (1.1) are bounded.

Corollary. In addition to the hypothesis of Theorem (II), suppose that assumption A_5 also holds and $\lim_{t \to \infty} a(t) = k > 0$; then all solutions of (1.1) and their derivatives are bounded.

Theorem (III). Suppose that assumptions A_1, A_2, A_3, and A_4 hold and in addition that $a(t) \geqq a_0 > 0$ for $t \geqq T$, and $\int^{\infty} |a'(t)| \, dt < \infty$. Then all solutions of (1.1) are bounded.

Corollary. In addition to the hypothesis of Theorem (III), suppose that assumption A_5 also holds; then all solutions of (1.1) and their derivatives are bounded.

The method of proof for the above results is based essentially on the well-known lemma of Gronwall [10], which is also known as the Bellman's lemma. In this paper, we use in addition to this fundamental lemma, its generalizations [11], [12], and techniques borrowed from Lyapunov's stability theory.

It might be of interest to note that quite a few results in [4] are incorrect; in particular Theorems 5 and 6. Also, Theorems 3 and 4 are stated incorrectly.

2. Boundedness Theorems I

Theorem 1. Suppose that assumptions A_1, A_2, A_3 and A_4 hold and that $a(t) > 0$ for $t \geqq T$ and there exists a non-negative function $\alpha(t)$ such that $-a'(t) \leqq \alpha(t) a(t)$ with $\int^{\infty} \alpha(s) \, ds < \infty$. Then all solutions of (1.1) are bounded.

Proof. Write equation (1.1) in its system form ($y_1 = u$):

$$\begin{cases} \dfrac{dy_1}{dt} = y_2 , \\ \\ \dfrac{dy_2}{dt} = -a(t) f(y_1) g(y_2) . \end{cases} \tag{2.1}$$

For system (2.1), we construct the following function:

$$V(t, y_1, y_2) = \int_0^{y_1} f(s)\, ds + \frac{1}{a(t)} \int_0^{y_2} \frac{s\, ds}{g(s)} \tag{2.2}$$

Clearly, under the hypothesis of the theorem, we have $V > 0$ whenever $y_1^2 + y_2^2 \neq 0$, and by A_4, $V \to \infty$ as $y_1 \to \infty$. Differentiating with respect to t, we obtain

$$V'(t, y_1, y_2) \leqq -\frac{a'(t)}{a^2(t)} \int_0^{y_2} \frac{s\, ds}{g(s)} \leqq \alpha(t)\, V(t, y_1, y_2) ,$$

hence,

$$V(t, y_1, y_2) \leqq V(T, y_1(T), \quad y_2(T)) \left\{ \exp \int_T^t \alpha(s)\, ds \right\} < \infty \tag{2.3}$$

for all t; and therefore all solutions of (1.1) are bounded. Furthermore, if assumption A_5 also holds and $a(t) \leqq a_1$ for all $t \geqq T$, then $u'(t)$ is also bounded for in this case $V \to \infty$ as $y_2 \to \infty$. Thus,

Corollary. In addition to the hypothesis of Theorem 4, suppose that assumption A_5 holds and $a(t) \leqq a_1$ for all $t \geqq T$; then all solutions and their derivatives are bounded.

Theorem 1 and its corollary generalize a result of Zhang ([4], Theorem 3). By taking $\alpha(s) \equiv 0$, Theorem 1 reduces to Theorem I. All these results are extensions of a theorem of Klokov ([13], Theorem 1). We remark that a slightly weaker version of Klokov's theorem may also be found in Waltman [14].

By the above result, we may conclude for example that all solutions $u(t)$ and their derivatives $u'(t)$ of the following equation:

$$u''(t) + (1 + e^{-t} \sin t)\, u^{\frac{3}{2}}(t) \left[2 + \cos u(t)\right] = 0$$

are bounded. On the other hand, no previously available result seems to yield such a conclusion.

Theorem 2. Suppose that assumptions A_1, A_2, A_3, and A_4 hold and that $a(t) > 0$, $a(t) \to 0$, and there exists a $\alpha(t) \geqq 0$ such that $a'(t) \leqq \alpha(t) a(t)$ while $\int^{\infty} \alpha(s)\, ds < \infty$. Then all solutions of (1.1) satisfy: $a(t) F(u(t)) = a(t) \int^{u(t)} f(s)\, ds < \infty$ [1]) for $t \geqq T$, and all its derivatives are bounded.

Proof. Consider the following function:

$$V(t, y_1, y_2) = a(t) \int_0^{y_1} f(s)\, ds + \int_0^{y_2} \frac{s\, ds}{g(s)} \tag{2.4}$$

and note that

$$\begin{aligned} V'(t, y_1, y_2) &\leqq a'(t) \int_0^{y_1} f(s)\, ds \\ &\leqq \alpha(t) a(t) \int_0^{y_1} f(s)\, ds \\ &\leqq \alpha(t) V(t, y_1, y_2). \end{aligned}$$

Hence again, we arrive at (2.3), from which the conclusion and the following corollary follow at once.

Corollary. Under the hypothesis of Theorem 5, if in addition assumption A_5 holds and $a(t) \geqq a_0 > 0$ for $t \geqq T$; then all solutions of (1.1) are also bounded.

By taking $\alpha(t) \equiv 0$, Theorem 2 and its corollary reduce to Theorem (II); similarly by taking $g(u') \equiv 1$, $f(u) = u^{2n-1}$ where n is a positive integer, the above result reduces to Theorem 4 of [4].

We now present two results on the boundedness of solutions by linear functions, i.e. $|u(t)| \leqq c|t|$ for some positive constant c, and for $t \geqq T$; and the existence of limit of $u'(t)$ as $t \to \infty$.

Theorem 3. Suppose that assumptions A_1, A_2 and A_3 hold and that

(i) $|f(u)| \leqq M |u|^{\alpha}$, where M, $\alpha > 0$,

(ii) $\int^{\infty} |a(s)| s^{\alpha}\, ds < \infty$,

(iii) $0 < g(v) \leqq K$ for all v;

then the derivative $u'(t)$ of any solution $u(t)$ of (1.1) has a limit if the initial conditions satisfy: for $\alpha > 1$,

$$\left\{ KM(\alpha - 1) \int_{t_0}^{\infty} s^{\alpha} |a(s)|\, ds \right\}^{\frac{1}{1-a}} \geqq \left\{ |u(t_0)| + |u'(t_0)| \right\} \tag{2.5}$$

---

1) $a(t) F(u(t)) < \infty$ means that a solution $u(t)$ of (1.1) is either bounded or unbounded, but in case of unboundedness must satisfy $a(t) F(u(t)) < \infty$. (Note that $a(t) \to O$, as $t \to \infty$ and assumption A_4.)

Proof. Consider equation (1.1) in its equivalent integral equation form:

$$u(t) = u(t_0) + u'(t_0)t - \int_{t_0}^{t} (t-s)\,a(s)f(u(s))g(u'(s))\,ds\,.$$

From the hypothesis of the theorem, we have for $t \geqq t_0 \geqq 1$ the following estimate:

$$\frac{|u(t)|}{t} \leqq (|u(t_0)| + |u'(t_0)|) + \int_{t_0}^{t} s^{\alpha} KM\,|a(s)|\left(\frac{|u(s)|}{s}\right)^{\alpha} ds \qquad (2.6)$$

By a variation of Gronwall's lemma (see e.g. [15], [16]), we obtain for $t \geqq t_0 \geqq 1$,

$$\frac{|u(t)|}{t} \leqq \left\{(|u(t_0)| + |u'(t_0)|)^{1-\alpha} + KM(1-\alpha)\int_{t_0}^{t} s^{\alpha}\,|a(s)|\,ds\right\}^{\frac{1}{1-\alpha}} \qquad (2.7)$$

which is finite on account of (2.5) and (i). Now from

$$u'(t) = u'(t_0) - \int_{t_0}^{t} a(s)f(u(s))g(u'(s))\,ds$$

and that

$$\left|\int_{t}^{t_0} a(s)f(u(s))g(u'(s))\,ds\right| \leqq MK\int_{t}^{t_0} |a(s)\,u^{\alpha}(s)|\,ds$$

$$\leqq MKC^{\alpha}\int_{t_0}^{t} |a(s)|\,s^{\alpha}\,ds$$

where C denotes the bound given in (2.7); we conclude that the limit $\lim_{t\to\infty} u'(t) = L$ exists.

We remark that the method of the above proof has also been applied by the author [17] to prove a generalization of a recent result of Waltman [18].

Theorem 4. Suppose that assumptions A_1, A_2 and A_3 hold and in addition that

(a) $|f(u)| \leqslant M\,h(|u|)$,

where $M > O$ aud $h(r)$ is a non-decreasing continous function such that $h(\lambda r) \leqq \lambda^{\alpha} h(r)$, where λ is positive and α is a positive constant; and

$$H(x) = \int_{\infty}^{x} \frac{dr}{h(r)} \to \infty \text{ as } x \to \infty,$$

(b) $\int |a(s)| s^{\alpha} ds < \infty$,

(c) $0 < g(v) \leqq K$ for all v;

then the derivative of any solution of (1.1) has a limit.

Proof. Proceeding as in the above proof, we obtain instead of (3.2) the following estimate:

$$\frac{|u(t)|}{t} \leqq (|u(t_0)| + |u'(t_0)|) + \int_{t_0}^{t} s^{\alpha} KM |a(s)| h\left(\frac{|u(s)|}{s}\right) ds ,$$

from which we conclude from a result of Bihari [14] that

$$\frac{|u(t)|}{t} \leqq H^{-1}\left(H(|u(t_0)| + |u'(t_0)|) + KM \int_{t_0}^{t} |a(s)| s^{\alpha} ds\right)$$

which is bounded for t on account of assumption *(a)*. The remaining proof follows verbatim that of Theorem 3.

3. Boundedness Theorems II

Theorem 5. Suppose that assumptions A_1, A_2, A_3 and A_4 hold and in addition that

(i) $a(t) > 0, \quad a'(t) \geqq 0, \quad$ for $\quad t \geqq T$,

(ii) $\frac{d}{dt}\left(\frac{b}{a}\right) \leqq \beta(t)\left(1 + \frac{b}{a}\right), \quad$ with $\quad \int^{\infty} \beta(s) ds < \infty$

and

$$\left(1 + \frac{b}{a}\right) \geqq \varepsilon > 0 ;$$

then every solution of (1.1) with $(a(t) + b(t))$ replacing $a(t)$ is bounded.

Proof. Make the following substitution for the independent variable, $x = \int^{t} \sqrt{a(s)}\, ds$ which tends to infinity as $t \to \infty$, and obtain instead of (1.1) its transformed equation:

$$\frac{d^2 u}{dx^2} + \frac{1}{2}\left(\frac{a}{a^{3/2}}\right)\frac{du}{dx} + \left(1 + \frac{b}{a}\right) f(u) g(u') = 0 \qquad (3.1)$$