# CONDITION FOR EXISTENCE OF A SMALLEST BOREL ALGEBRA CONTAINING A GIVEN COLLECTION OF SETS

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Objekttyp: **Article**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **13 (1967)**

Heft 1: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **29.03.2024**

Persistenter Link: https://doi.org/10.5169/seals-41532

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# A CONDITION FOR EXISTENCE OF A SMALLEST BOREL ALGEBRA CONTAINING A GIVEN COLLECTION OF SETS

by Arthur B. BROWN and Gerald FREILICH

The origin of this note lies in an oversight appearing in [1] and [2], a difficulty that was already realized by the translators of [1]. (See Translators' Note in [1], page 16. Since situations arise in which $B$-algebras with different units are used, Theorem 4 on page 19 of [1] and on page 25 of [2] requires an additional hypothesis. See the theorem below.) It is hoped that the present note will be of independent interest.

DEFINITIONS. *A $\sigma$-ring (of sets) is a non-empty collection of sets closed under the operations of difference (of a pair of sets) and countable union.*

*A Borel algebra, or* B*-algebra, is a $\sigma$-ring which has an element that contains every other element of the $\sigma$-ring. The (unique) maximal element is called the unit of the* B*-algebra.*

*A member of a collection of sets is called the smallest member if it is contained in every other member of the collection.*

LEMMA. *If* S *is a non-empty collection of sets each contained in a set* X, *then there exists a smallest* B*-algebra* B (S) *with unit* X *containing* S.

*Proof.* Take $B(S)$ to be the intersection of all $B$-algebras with unit $X$ that contain $S$.

If we want now to generalize the lemma by omitting the requirement that the $B$-algebras under consideration have the same unit $X$, an additional hypothesis is necessary.

THEOREM. *Let* S *be a non-empty collection of sets whose union is* X. *Then there is a smallest* B*-algebra containing* S *if and only if* X *is the union of some countable collection of sets of* S. *If there is a smallest* B*-algebra containing* S, *then that algebra has unit* X *and is the algebra* B (S) *of the Lemma.*

*Proof.* Suppose that $X$ is the union of a countable collection of sets of $S$. Let $W$ be any $B$-algebra containing $S$, where sets of W are not restricted to be subsets of $X$, and let $D = W \cap B(S)$, where $B(S)$ is the smallest

$B$-algebra with unit $X$ and containing $S$. (See Lemma.) Since $W$ is a $\sigma$-ring, $X \in W$; hence $X \in D$. Since $D \subseteq B(S)$, the sets in $D$ are subsets of $X$. Since $W$ and $B(S)$ are $\sigma$-rings, so is $D$. Hence $D$ is a $B$-algebra with unit $X$. Since $B(S)$ is the smallest $B$-algebra with unit $X$, we infer that $B(S) \subseteq D$, and hence $B(S) = D$. Thus $B(S) = D \subseteq W$, so $B(S)$ is the smallest $B$-algebra containing $S$, as was to be proved.

Now suppose $X$ is not the union of any countable collection of sets of $S$. Choose $\alpha \notin X$ and let $Y = X \cup \{\alpha\}$. Let $S' = \{A : A \subseteq \cup_{i=1}^{\infty} S_i, S_i \in S\}$, $S'' = \{A : A \in S' \text{ or } (Y-A) \in S'\}$. Then any subset of a member of $S'$ is a member of $S'$, and $S'$ is clearly a $\sigma$-ring. It is obvious that $S''$ contains $S$. We now prove that $S''$ is a $B$-algebra.

Since $\varnothing \in S'$, $Y \in S''$, so $Y$ will be the unit. Let $A_1, A_2, \ldots$ be members of $S''$. If each $A_j \in S'$, with $A_j \subseteq \cup_{i=1}^{\infty} S_{ij}$, $S_{ij} \in S$, then $\cup_{j=1}^{\infty} A_j \subseteq \cup_{j=1}^{\infty} \cup_{i=1}^{\infty} S_{ij}$, so $\cup_{j=1}^{\infty} S_j \in S' \subseteq S''$. If some $A_k \notin S'$, then $Y-A_k \in S'$. Hence $Y-\cup_{j=1}^{\infty} A_j = \cap_{j=1}^{\infty} (Y-A_j) \subseteq Y-A_k \in S'$, so that $Y-\cup_{j=1}^{\infty} A_j \in S'$ and consequently $\cup_{j=1}^{\infty} A_j \in S''$. Thus it is proved that $S''$ is closed under countable unions. We now consider differences.

Suppose $\{A, B\} \subseteq S''$. If $A \in S'$ then $A-B \subseteq A \in S'$, so $A-B \in S'$ and hence $A-B \in S''$. If $A \notin S'$ then $Y-A \in S'$, and if $B \in S'$ we have $Y-(A-B) \subseteq (Y-A) \cup B \in S'$, so $Y-(A-B) \in S'$; hence $A-B \in S''$. If $A \notin S'$ and $B \notin S'$, then $A-B = (Y-B) - (Y-A) \in S' \subseteq S''$. This completes the proof that $S''$ is a $B$-algebra.

Since $X$ is not the union of any countable collection of sets of $S$, it is clear that $X \notin S'$. Consequently $X \notin S''$, for if $X = Y-A$ with $A \in S'$, we would have $\alpha \in X$, contrary to the choice of $\alpha$. We are now in a position to complete the proof.

If there were a smallest $B$-algebra $V$ containing $S$, then by the definition of unit, the unit $E$ of $V$ would contain $X$. Furthermore, $V$ would be contained in the set of all subsets of $X$ (the latter being a $B$-algebra containing $S$), so $E \subseteq X$. Hence $X$ would be the unit $E$ of $V$. Then, from $X \in V$ and $X \notin S''$, it would follow that $V \nsubseteq S''$, contrary to the fact that $S''$ is a $B$-algebra containing $S$. Hence there is no smallest $B$-algebra containing $S$.

EXAMPLES. Let $X$ be an uncountable set and let $S$ be the set of all countable subsets of $X$. By the Theorem, there is no smallest $B$-algebra containing $S$.

Note that the Theorem implies that if $S$ is a non-empty collection of sets such that there is no smallest $B$-algebra containing $S$, then the union of the members of $S$ must be uncountable.

## REFERENCES

1. KOLMOGOROV, A. N. and S. V. FOMIN, *Elements of the Theory of Functions and Functional Analysis*, Vol. 2 (translated by H. Kamel and H. Komm), Graylock Press, Albany, N. Y., 1961. Translators' Note, page 16: " This definition leads to difficulties in the statements and proofs of certain theorems in the sequel... "

2. —— and S. V. FOMIN, *Measure, Lebesgue Integrals, and Hilbert Space* (translated by N. A. Brunswick and A. Jeffrey), Academic Press, New York, 1961.

*(Reçu le 20 décembre 1966)*

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