# SOME APPLICATIONS OF THE GAUSS-LUCAS THEOREM


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# SOME APPLICATIONS OF THE GAUSS-LUCAS THEOREM [1]

par L. A. RUBEL

The Gauss-Lucas Theorem, that the zeros of the derivative of a non-constant polynomial lie in the convex hull of the set of zeros of the polynomial, is a surprisingly powerful and versatile tool in classical analysis, despite its simplicity. To illustrate this point, we prove several results, using the Gauss-Lucas Theorem as our principal tool. To begin with, we present a short proof of the Gauss-Lucas Theorem. As applications, we first give a new lower bound on the largest modulus of the zeros of a polynomial, in terms of the coefficients of the polynomial. Next, we strengthen slightly a result of Edrei [2] on the zeros of the partial sums of a power series. Finally, we reformulate a method of Fejér, and use it to strengthen some classical results on lacunary polynomials and entire functions with lacunary power series.

THE GAUSS-LUCAS THEOREM. *The zeros of the derivative of a non-constant polynomial* P *lie in the convex hull of the set of zeros of* P.

*Proof.* It is enough to prove that any open half-plane that contains the zeros of $P$ contains the zeros of $P'/P$. Without loss of generality, we may suppose that all the zeros of $P$ lie in the open right half-plane. Writing.

$$P(z) = a \prod (z - z_n) ,$$

with $z_n = x_n + iy_n$, $x_n > 0$, we have

$$P'(z)/P(z) = \sum (z - z_n)^{-1} ,$$

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so that

$$Re\left(P'(z)/P(z)\right) = \sum (x - x_n) \mid z - z_n \mid^{-2} .$$

Hence, if $x < 0$, then $Re\,(P'(z)/P(z)) < 0$, and $P'/P$ consequently has no zeros in the open left half-plane.

THEOREM. *If* $P(z) = a_0 + a_1 z + \dots + a_n z^n$, $a_n \neq 0$, then $P$ *has a zero of modulus at least*

$$\max_{0 \leq \nu \leq n-1} \left\{ \binom{n}{\nu}^{-1} \left| \frac{a_{n-\nu}}{a_n} \right| \right\}^{1/\nu},$$

*Proof.* First, considering the $\nu$-th derivative of $P$,

$$P^{(\nu)}(z) = \sum_{k=\nu}^{n} a_k \frac{k!}{(k-\nu)!} z^{k-\nu},$$

we see that the product of the zeros of $P^{(\nu)}$ is

$$\pm \frac{a_\nu}{a_n} \frac{\nu!\,(n-\nu)!}{n!},$$

so that $P^{(\nu)}$ must have a zero of modulus at least

$$\left| \frac{a_\nu}{a_n} \frac{\nu!\,(n-\nu)!}{n!} \right|^{1/(n-\nu)},$$

since there are $n - \nu$ roots. But by the Gauss-Lucas Theorem, the modulus of the largest root of $P$ cannot be smaller than this, and the result is proved on interchanging $\nu$ and $n - \nu$.

COROLLARY. *If* $Q(z) = b_0 + b_1 z + \dots + b_n z^n$, $b_0 \neq 0$, then $Q$ *has a zero of modulus at most*

$$\min_{0 \leq \nu \leq n-1} \left\{ \binom{n}{\nu} \left| \frac{b_0}{b_\nu} \right| \right\}^{1/\nu}.$$

*Proof.* Apply the preceding theorem to $P(z) = z^n Q(1/z)$.

The first part of the next result was proved, in a different way, by Edrei [2].

THEOREM. *Suppose that the formal power series, not a polynomial,*

$$f(z) = a_0 + a_1 z + a_2 z^2 + \dots , \qquad a_0 \neq 0 ,$$

*has the property that for infinitely many $n$, there is a closed half-plane $T_n$ that contains the origin and that contains all the zeros of the partial sum $S_n(z) = a_0 + a_1 z + \dots + a_n z^n$. Then no two consecutive coefficients of $f(z)$ may vanish. If one coefficient vanishes, then there is a line through the origin that contains all the zeros of all the partial sums.*

*Proof.* Suppose, by way of contradiction, that two consecutive coefficients of $f$ do vanish. They are contained in a block of zero coefficients, flanked left and right by non-zero coefficients, say $a_p$ and $a_q$, respectively. Choose $n \geqq q$, and differentiate $S_n$ successively $p$ times, to get $S_n^* = S_n^{(p)}$,

$$S_n^*(z) = a_p^* + a_q^* z^r + z^{r+1} R(z) ,$$

where $a_p^* \neq 0$, $a_q^* \neq 0$, $r = q - p$, $R$ is a polynomial, and $S_n^*$ has degree $n - p$. Now define $S_n^{**}$ by $S_n^{**}(z) = z^{n-p} S_n^*(1/z)$, so that $S_n^{**}$ is again a polynomial. Differentiate $S_n^{**}$ successively $m$ times, where $m = n - q$, to get a polynomial $S_n^{***}$,

$$S_n^{***}(z) = a_q^{**} + a_p^{**} z^s ,$$

where $a_q^{**} \neq 0$, $a_p^{**} \neq 0$, and $s = q - p \geqq 3$. Now if $S_n$ has all its zeros in a closed half-plane $T_n$ that contains the origin, then repeated applications of the Gauss-Lucas Theorem show that $S_n^*$ also has all its zeros in $T_n$. Then $S_n^{**}$ has all its zeros in the closed half-plane $T_n^* = \{1/z : z \varepsilon T_n\} \cup \{0\}$. Again repeatedly applying the Gauss-Lucas Theorem, we see that $S_n^{***}$ has all its zeros in $T_n^*$. But the zeros of $S_n^{***}$ are just $s$-th roots of $- a_q^{**}/a_p^{**}$, and since $s \geqq 3$, we have a contradiction.

In the sequel, the word « set » will denote subsets of the finite complex plane.

DEFINITION. *If $P$ is a polynomial, then $Z(P)$ denotes the set of zeros of $P$.*

**Definition.** *If $E$ is a finite set, then $K(E)$ denotes the convex hull of $E$, and $K^*(E)$ denotes $K(E) \cup \{0\}$.*

**Definition.** *If $E$ is a set, then $1/E$ denotes the set*

$$1/E = \{1/z : z \in E, \ z \neq 0\} .$$

**Definition:** *If $P$ is a polynomial,*

$$P(z) = a_0 + a_1 z + \dots + a_n z^n , \qquad n > 0 ,$$

*then $P^{\#}$ denotes the polynomial*

$$P^{\#}(z) = \frac{1}{n}\{a_0 + (a_0 + a_1 z) + \dots + (a_0 + a_1 z + \dots + a_{n-1} z^{n-1})\} ,$$

*that is,*

$$P^{\#}(z) = a_0 + \frac{n-1}{n} a_1 z + \frac{n-2}{n} a_2 z^2 + \dots + \frac{1}{n} a_{n-1} z^{n-1} .$$

In other words, $P^{\#}$ is just the arithmetic mean of the proper partial sums of $P$. The next result is latent in the paper of Fejér [3].

**Theorem.** *If $a_0 \neq 0$ and $a_n \neq 0$, then*

$$Z(P^{\#}) \subseteq \frac{1}{K\left(\dfrac{1}{Z(P)}\right)} ,$$

*or equivalently,*

$$K\left(\frac{1}{Z(P^{\#})}\right) \subseteq K\left(\frac{1}{Z(P)}\right) .$$

*Addendum.* It will be clear from the proof that if $a_0 \neq 0$, then $Z(P^{\#}) \subseteq 1/K^*(1/Z(P))$. Further, $Z(P^{\#}) \subseteq \{0\} \cup 1/K(1/Z(P))$ if $a_n \neq 0$ and $P(z) \neq a_n z^n$. In any event, so long as $P(z) \neq a_n z^n$, $Z(P^{\#}) \subseteq \{0\} \cup 1/K^*(1/Z(P))$.

*Proof.* A straightforward computation shows that if

$$R(z) = z^n P\left(\frac{1}{z}\right) = a_n + a_{n-1} z + \dots + a_0 z^n .$$

and

$$Q(z) = \frac{d}{dz} R(z) = a_{n-1} + 2a_{n-2} z + \dots + na_0 z^{n-1} ,$$

then

$$P^{\#}(z) = \frac{1}{n} z^{n-1} Q \frac{1}{z} = a_0 + \frac{n-1}{n} a_1 z + \dots + \frac{1}{n} a_{n-1} z^{n-1} .$$

Hence

$$Z(P^{\#}) \subseteq \frac{1}{Z(Q)} \quad \textit{if} \quad a_0 \neq 0 ,$$

while

$$Z(P^{\#}) \subseteq \{0\} \cup \frac{1}{Z(Q)} \quad \textit{if} \quad a_0 = 0 .$$

Since $R \neq$ const., we may apply the Gauss-Lucas Theorem to get

$$Z(Q) \subseteq K(Z(R)) .$$

Now

$$Z(R) \subseteq \frac{1}{Z(P)} \quad \textit{if} \quad a_n \neq 0$$

while

$$Z(R) \subseteq \{0\} \cup \frac{1}{Z(P)} \quad \textit{if} \quad a_n = 0 .$$

Combining these results, the theorem is proved.

COROLLARY. *If a disc with center at the origin is free of zeros of $P$, then it is free of zeros of $P^{\#}$.*

COROLLARY. *If $a_0 a_n \neq 0$ and if $P^{\#}$ has at least three zeros whose reciprocals are non-collinear, then so does $P$.*

The first corollary is the basis of the proofs of the next results. These results are quite classical, except that the first is somewhat elaborated.

THEOREM. *Suppose that*

$$P(z) = a_0 + a_1 z^{k_1} + a_2 z^{k_2} + \dots + a_n z^{k_n}$$

*where* $0 < k_1 < k_2 < \dots < k_n$, *and* $a_j \neq 0$ *for* $j = 0,1 \dots, n$.

*Let*

$$A = \frac{a_0}{a_1} \frac{k_2}{k_2 - k_1} \frac{k_3}{k_3 - k_1} \cdots \frac{k_n}{k_n - k_1}.$$

*Then P has at least one zero in the disc*

$$|z| \leqq |A|^{-1/k_1}.$$

*If* $k_1 \geqq 3$, *then P must have at least two distinct zeros in this disc, and at least three distinct zeros whose reciprocals lie on or outside the regular polygon whose vertices are the* $k_1$*-th roots of* $-1/A$.

*Proof.* Apply the operation $\#$ repeatedly, taking into account the degrees of the resulting polynomials, to get the polynomial

$$P^*(z) = a_0 + \frac{k_2 - k_1}{k_2} \frac{k_3 - k_1}{k_3} \cdots \frac{k_n - k_1}{k_n} a_1 z^{k_1}.$$

Applying the first corollary, we obtain the first part of the theorem, since the zeros of $P^*$ are the roots of $z^{k_1} = -A$. The other parts follow from simple geometric considerations and the fact that

$$K\left(\frac{1}{Z(P^*)}\right) \subseteq K\left(\frac{1}{Z(P)}\right).$$

THEOREM. *Suppose that f is a transcendental entire function with power series expansion*

$$f(z) = \sum_{k=0}^{\infty} a_k z^{n_k}, \quad a_k \neq 0 \quad for \quad k = 0, 1, 2, \dots,$$

*and suppose that*

$$\sum_{n_k > 0} \frac{1}{n_k} < \infty.$$

*Then the range of f contains each complex number.*

*Proof.* It is enough to prove that $f$ has a zero, and we may clearly suppose that $n_0 = 0$. From the proceding result, we see that the $n_k$-th partial sum of the power series for $f$ has a zero in the disc

$$|z| \leq \left\{ \frac{|a_0|}{|a_1|} \frac{1}{\left(1-\frac{n_1}{n_2}\right)\left(1-\frac{n_1}{n_3}\right)\cdots\left(1-\frac{n_1}{n_k}\right)} \right\}^{1/n_k}.$$

But since $\sum 1/n_k < \infty$, the product $\prod_2^\infty (1-(n_1/n_k))$ converges, so that there is a fixed disc with center at the origin that contains a zero of the $n_k$-th partial sum for $k = 2, 3, 4, \ldots$. It follows that $f$ has a zero in this disc, and the result is proved.

It should be pointed out that Biernacki [1] proved, under the same hypotheses, and using a stronger form of the Gauss-Lucas Theorem, that $f$ takes each complex value infinitely often. It is likely that our method can give a slight improvement of the preceding result, but not to the full strength of Biernacki's result. A recent result of G. and M. Weiss [5] gives a partial analogue for functions regular in the unit disc.

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