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# ON ERMAKOF'S CONVERGENCE CRITERIA AND ABEL'S FUNCTIONAL EQUATION. *)

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## I. Introduction

1. We owe to V. Ermakof ([1], [2]) very remarkable criteria for the convergence or divergence of infinite series $\Sigma\, f(\nu)$ ($f(x) > 0$) which uses the quotient

$$\frac{f(\Psi(x))\, \Psi'(x)}{f(x)} \tag{1}$$

for continuously differentiable function $\Psi(x)$ with the properties

$$\Psi(x) > x,\ \Psi(x) \to \infty\ (x \to \infty)\,.$$

As a matter of fact, the first discussion given by Ermakof [1] only established directly the connection with the convergence or the divergence of the integral

$$\int^{\infty} f(x)\, dx \tag{2}$$

so that in order to obtain the results concerning the infinite series we have to assume that $f(x)$ is monotonically decreasing or to make some analogous assumptions to permit the transition from the integral to the infinite series. We discuss some conditions of this kind in the sections 33-38.

2. In his second paper [2] Ermakof developped however a new and very ingenious method of proof using Abel's functional equation

$$\varphi(\Psi(x)) = \varphi(x) + 1\,.$$

*) This investigation was carried out under the contract DA-91-591-EUC-2824 of the Institute of Mathematics, University of Basle, with the US Department of the Army.

This method allows, under suitable regularity conditions on $\Psi(x)$, to connect directly the behavior of (1) with the convergence or divergence of the infinite series $\Sigma f(\nu)$, *without any monotony condition for* $f(x)$.[1])

But Ermakof only sketched his discussion and indicated as the sufficient additional condition to impose on $\Psi(x)$ that, in our notations, $\Psi'(x_0)$ is $\geqq 1$, for a suitable $x_0$.[2])

It appears however that this additional condition is not sufficient to carry the discussion through. In a paper [5], published 1955, I showed that if beyond Ermakof's condition $\Psi'(x)$ is supposed monotonically increasing, the method can be carried through, indeed. If on the other hand $\Psi'(x)$ is supposed monotonically decreasing the method worked but Ermakof's additional condition was not necessary.

3. In this communication I develop a new method of proof which allows to avoid Abel's functional equation and to obtain the essential results for not necessarily monotonic $f(x)$. This gives a direct and very elementary way of proof as well for monotonically increasing as, (in the case of convergence), for monotonically decreasing $\Psi'(x)$. Beyond that, this method allows also to prove the convergence criteria in the case that $\lim_{x \to \infty} \Psi'(x)$ exists and is finite (Theorems 4-6).

4. As to the divergence criterion, here too, a new result in the case of monotonically decreasing $\Psi'(x)$ can be obtained (Theorem 7), however, with a different method which has more points of contact with Ermakof's second proof — here we have to form a minorant of $f(x)$, which can be interpreted as the derivative of a solution of Abel's functional equation —.

5. In the first sections of this paper we give 3 Theorems concerning the convergence and divergence of the integral (2) generalizing some results given in our first paper [5]. Finally, in the last part of the paper we discuss Pringsheim's treatment

1 Curiously enough, Abel's functional equation was also treated by Korkine in the note [4] where he gave another and direct proof of Ermakof's criteria for monotonic $f(x)$, without using, however, this functional equation.

2) Ermakof says in his paper [2] in a footnote on p. 142: " C'est la seule condition pour que notre démonstration soit juste."

of the problem and prove generalized versions of Pringsheim's results.

This note brings therefore an improvement and simplification of the sections I-V and XI of [5], while I have nothing to add to the sections VI-X of [5].

## II. Ermakof's direct Method

6. The form of the expression (1) makes it plausible that we will have to use the integral transformation formula

$$\int_a^b f(\Psi(x))\,\Psi'(x)\,dx = \int_{\Psi(a)}^{\Psi(b)} f(x)\,dx. \tag{3}$$

In order to be able to use (3) we have in any case to assume that $f(x)$ is integrable in the integration interval and $\Psi(x)$ totally continuous between $a$ and $b$. However, additional conditions are necessary and two such conditions are known either of which ensures the relation (3):

$J_1$: $|f(x)|$ is uniformly bounded in the integration interval;
$J_2$: $\Psi(x)$ is monotonically increasing or monotonically decreasing.

7. Theorem 1. *Assume that $\psi(x)$ and $\Psi(x)$ are totally continuous for $x \geqq x_0$ and that we have for a sequence $b_\nu \geqq x_0$ ($\nu = 1,2, \ldots$)*

$$\psi(b_\nu) \leqq \Psi(b_\nu),\ \Psi(b_\nu) \to \infty\ (\nu \to \infty). \tag{4}$$

*Let $f(x)$ be $\geqq 0$ on no half-line $x \geqq \xi$ almost everywhere $= 0$, and measurable in an interval $J$ containing all values of $\psi(x)$ and $\Psi(x)$ for $x \geqq x_0$. Assume further that for any finite subinterval of $J$ the transformation formula (3) holds as well for $\psi(x)$ as for $\Psi(x)$. Then, if we have for almost all $x$ with $x \geqq x_0$ and for an $\alpha$ with $0 < \alpha < 1$:*

$$f(\Psi(x))\,\Psi'(x) \leqq \alpha f(\psi(x))\,\psi'(x)\ (x \geqq x_0),\ 0 < \alpha < 1, \tag{5}$$

*the integral (2) is convergent and we have for all $x \geqq x_0$:*

$$\Psi(x) > \psi(x) \quad (x \geqq x_0). \tag{6}$$

8. *Proof*. For an arbitrary $x \geqq x_0$ integrate (5) between $x$ and $b_\nu > x$. Then we have, using (3):

$$\int_{\Psi(x)}^{\Psi(b_\nu)} f(x)\,dx \leqq \alpha \int_{\psi(x)}^{\psi(b_\nu)} f(x)\,dx$$

and this remains true, by (4), if $\psi(b_\nu)$ is replaced by $\Psi(b_\nu)$. We can therefore write

$$\int_{\Psi(x)}^{\Psi(b_\nu)} f(x)\,dx \leqq \alpha \int_{\Psi(x)}^{\Psi(b_\nu)} f(x)\,dx + \alpha \int_{\psi(x)}^{\Psi(x)} f(x)\,dx\,,$$

or, bringing the first right hand term to the left:

$$(1-\alpha) \int_{\Psi(x)}^{\Psi(b_\nu)} f(x)\,dx \leqq \alpha \int_{\psi(x)}^{\Psi(x)} f(x)\,dx\,.$$

But here, if we take $x = b_1$ it follows for $b_\nu \to \infty$ the convergence of (2) and also that the right hand expression is $> 0$ for any $x \geqq x_0$. (6) follows immediately and the Theorem 1 is proved.

9. Theorem 2. *Assume that $\psi(x)$, $\Psi(x)$ are totally continuous for $x \geqq x_0$ and that $f(x)$ is non-negative and measurable in an interval $J$ containing all values of $\psi(x)$ and $\Psi(x)$. Assume that (3) holds as well for $\psi(x)$ as for $\Psi(x)$. Assume further that there exists an $a \geqq x_0$ such that:*

$$\int_{\psi(a)}^{\Psi(a)} f(x)\,dx > 0\,, \tag{7}$$

*and a sequence $b_\nu \geqq x_0$ $(\nu = 1,2...)$ such that:*

$$\psi(b_\nu) \to \infty\,,\quad \Psi(b_\nu) \to \infty\ (\nu \to \infty)\,. \tag{8}$$

*Then if we have for almost all $x \geqq x_0$:*

$$f(\Psi(x))\,\Psi'(x) \geqq f(\psi(x))\,\psi'(x)\,, \tag{9}$$

*the integral (2) is divergent and we have for all $x \geqq a$:*

$$\Psi(x) > \psi(x) \quad (x \geqq a)\,. \tag{10}$$

10. *Proof.* For any $x > a$ we obtain from (9), integrating on both sides from $a$ to $x$ and using (3):

$$\int_{\Psi(a)}^{\Psi(x)} f(x)\,dx \geqq \int_{\psi(a)}^{\psi(x)} f(x)\,dx$$

and therefore

$$\int_{\psi(x)}^{\Psi(x)} f(x)\,dx \geqq \int_{\psi(a)}^{\Psi(a)} f(x)\,dx \quad (x \geqq a)\,. \tag{11}$$

This proves already (10).

Putting in (11) $x = b_\nu$ it follows

$$\int_{\psi(b_\nu)}^{\Psi(b_\nu)} f(x)\,dx \geqq \int_{\psi(a)}^{\Psi(a)} f(x)\,dx \tag{12}$$

while, if (2) were convergent, the left side integral in (12) would tend to 0.

Theorem 2 is proved.

11. THEOREM 3. *Assume that $\psi(x)$ and $\Psi(x)$ are totally continuous for $x \geqq x_0$, that (3) holds as well for $\psi(x)$ as for $\Psi(x)$ and $f(x)$ is $\geqq 0$ and measurable in an interval containing all values of $\psi(x)$ and $\Psi(x)$ for $x > x_0$ without being almost everywhere $= 0$ in $(\Psi(a), \infty)$. Assume further that there exists a constant $\gamma$, $0 < \gamma < 1$, and a sequence $b_\nu \geqq x_0$ $(\nu = 1,2,\ldots)$ such that*

$$\gamma\psi(b_\nu) \leqq \Psi(b_\nu)\,,\ 0 < \gamma < 1\,,\ \psi(b_\nu) \to \infty\ (\nu \to \infty)\,, \tag{13}$$

*and further that for a constant $c$ from a certain $x = x_1 \geqq x_0$ on:*

$$f(x) \leqq \frac{c}{x} \quad (x \geqq x_1)\,. \tag{14}$$

*Assume finally that for a constant $\alpha$, $0 < \alpha < 1$:*

$$f(\Psi(x))\,\Psi'(x) \leqq \alpha f(\psi(x))\,\psi'(x), \quad 0 < \alpha < 1\,. \tag{15}$$

*Then the integral (2) converges and we have $\Psi(x) > \psi(x)$ for all $x > x_0$.*

12. *Proof.* We have as in the proof of the Theorem 1:

$$\int_{\Psi(x_0)}^{\Psi(b_\nu)} f(x)\,dx \leqq \alpha \int_{\psi(x_0)}^{\psi(b_\nu)} f(x)\,dx$$

and therefore, using (13)

$$\int_{\Psi(x_0)}^{\gamma\psi(b_\nu)} f(x)\,dx \leqq \alpha \int_{\psi(x_0)}^{\psi(b_\nu)} f(x)\,dx = \alpha \int_{\psi(x_0)}^{\gamma\psi(b_\nu)} f(x)\,dx + \alpha \int_{\gamma\psi(b_\nu)}^{\psi(b_\nu)} f(x)\,dx\,.$$

But the last right hand integral is, by (14), $\leqq c \log\frac{1}{\gamma}$, so that we obtain:

$$(1-\alpha) \int_{\Psi(x_0)}^{\gamma\psi(b_\nu)} f(x)\,dx \leqq \int_{\psi(x_0)}^{\Psi(x_0)} f(x)\,dx + c\,\log\frac{1}{\gamma}.$$

The convergence of (2) follows now immediately from $\psi(b_\nu) \to \infty$.

13. Suppose that we have, on the other hand, for an $a > x_0$:

$$\Psi(a) \leqq \psi(a)\,.$$

Proceeding then as in the proof of the Theorem 1 we have, as from $\psi(b_\nu) \to \infty$ and the total continuity of $\psi(x)$ follows $b_\nu \to \infty$, for $b_\nu \geqq a$:

$$\int_{\Psi(a)}^{\Psi(b_\nu)} f(x)\,dx \leqq \alpha \int_{\psi(a)}^{\psi(b_\nu)} f(x)\,dx\,,$$

and, for $\nu \to \infty$:

$$\int_{\Psi(a)}^{\infty} f(x)\,dx \leqq \alpha \int_{\psi(a)}^{\infty} f(x)\,dx\,.$$

But here the left hand integral is $> 0$, the right hand integral is majorized by it and the relation is impossible for $\alpha < 1$. [3])

## III. A new method for not necessarily monotonic $f(x)$

14. Theorem 4. *Assume that $\Psi(x)$ is for $x \geqq x_0$ a positive and monotonically increasing differentiable function for which*

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3) Observe that in Ermakof's paper [1] the criteria are given in the following form: $\sum^{\infty} f(\nu)$ for a monotonic $f(x)$ is convergent or divergent according as

$$\lim_{x\to\infty} \frac{f(\Psi(x))\Psi'(x)}{f(\psi(x))\psi'(x)}$$

is $<1$ or $>1$. In the note [2] Ermakof takes $\Psi(x) \equiv x$ which is no essential specialisation. However, the conditions (5) for convergence and (9) for divergence (with the specialisation $\Psi(x) \equiv x$) are already found in the textbooks, see e.g. [3].

*$\Psi'(x)$ is also monotonically increasing and that we have:*

$$\Psi(x) > x \quad (x \geqq x_0). \tag{16}$$

*Suppose further that $f(x)$ is $> 0$ for $x \geqq x_0$ and integrable and bounded from below by a positive number in any finite subinterval of $\langle x_0, \infty)$. If we have for all $x \geqq x_0$:*

$$f(\Psi(x))\,\Psi'(x) \geqq f(x), \tag{17}$$

*the sum*

$$\sum_{\nu \geqq x_0} f(\nu) \tag{18}$$

*is divergent.*

15. *Proof.* Introduce the function

$$F(x) = \operatorname*{Inf}_{x_0 \leqq u \leqq x} f(u); \tag{19}$$

then $F(x)$ is monotonically decreasing and we have for each $x \geqq x_0$:

$$F(x) = \lim_{\kappa \to \infty} f(u_\kappa)$$

for a convenient sequence $u_\kappa$ from the interval $\langle x_0, x\rangle$.

We can write therefore for a certain sequence $v_\kappa$ from the interval $\langle x_0, x\rangle$:

$$F(\Psi(x))\,\Psi'(x) = \lim_{\kappa \to \infty} f(\Psi(v_\kappa))\,\Psi'(x) \geqq \overline{\lim_{\kappa \to \infty}}\, f(\Psi(v_\kappa))\,\Psi'(v_\kappa).$$

This is, however, by (17) $\geqq \overline{\lim}\, f(v_\kappa) \geqq F(x)$.

It follows

$$F(\Psi(x))\,\Psi'(x) \geqq F(x),$$

so that the integral $\int^{\infty} F(x)\,dx$ is divergent. Since $F(x)$ is monotonic, the same follows for the series $\sum^{\infty} F(\nu)$ which has (18) as a majorant. The Theorem 4 is proved.

16. Theorem 5. *Assume that $\Psi(x)$ is for $x \geqq x_0$ a positive and monotonically increasing differentiable function for which (16) holds. Assume further that $\Psi'(x)$ is either, from a certain $x$ on, monotonically increasing or, for $x \to \infty$, convergent to a finite*

*limit* $\omega$. *Assume finally that* $f(x)$ *is* $\geqq 0$ *for* $x \geqq x_0$, *measurable and bounded in each interval* $x_0 \leqq x \leqq a$ *and satisfies for all* $x \geqq x_0$ *and for a certain constant* $\delta < 1$ *the inequality:*

$$f(\Psi(x))\,\Psi'(x) \leqq \delta f(x) \quad (x \geqq x_0). \tag{20}$$

*Then the series (18) is convergent.*

17. *Proof*. Take a number $\beta$ with $1 > \beta > \delta$. Observe that $\Psi'(x)$ certainly cannot have for $x \to \infty$ a limit $\omega < 1$. For otherwise we would have, with $x \to \infty$,

$$(\Psi(x) - x)' \to \omega - 1 < 0, \quad \Psi(x) - x \to -\infty$$

contrary to (16).

We have therefore in any case, from a certain $x$ on, $\Psi'(x) \geqq \delta$, and, by (20), $f(\Psi(x)) \leqq f(x)$. We can therefore assume, changing $x_0$ if necessary, that we have:

$$f(\Psi(x)) \leqq f(x) \quad (x \geqq x_0). \tag{21}$$

Further, if we have $\Psi(x) \to \omega \geqq 1$ and if $\omega$ is finite there certainly exists an $x_1$ such that we have, if $x \geqq x_1$, $y \geqq x_1$,

$$\frac{\delta}{\beta} \leqq \frac{\Psi'(x)}{\Psi'(y)} \leqq \frac{\beta}{\delta}.$$

We can therefore assume, increasing $x_0$ if necessary, that we have:

$$\Psi'(x) \leqq \frac{\beta}{\delta}\Psi'(y) \quad (y \geqq x \geqq x_0), \tag{22}$$

and this is obviously also true if $\Psi'(x)$ is monotonically increasing, so that we can now assume (22) as being true under the conditions of our Theorem.

18. Put

$$x_0 = a_0, \ \Psi(a_0) = a_1, \ldots, \Psi(a_\nu) = a_{\nu+1}, \ldots$$

The sequence $a_\nu$ is monotonically increasing. If $\lim a_\nu = \tau$ were finite, we would have $\Psi(\tau) = \tau$, contrary to (16). Therefore we have $a_\nu \uparrow \infty$.

We have therefore for any $x \geqq x_0$ an index $\nu$ such that $a_\nu \leqq x < a_{\nu+1}$.

Denoting by $c$ an upper bound for $f(x)$ in the interval $\langle a_0, a_1)$ it follows then from (21):

$$f(x) \leqq c \quad (x \geqq x_0).$$

19. Put

$$G(x) = \operatorname*{Sup}_{u \geqq x} f(u). \tag{23}$$

$G(x)$ is finite and monotonically decreasing and we have:

$$f(x) \leqq G(x) \quad (x \geqq x_0). \tag{24}$$

By (23), there exists for any $x \geqq x_0$ a sequence of numbers $u_\kappa$, $u_\kappa \geqq x$ such that $G(\Psi(x)) = \lim_{\kappa \to \infty} f(\Psi(u_\kappa))$ and by (22)

$$G(\Psi(x))\,\Psi'(x) = \lim_{\kappa \to \infty} f(\Psi(u_\kappa))\,\Psi'(x) \leq \overline{\lim_{\kappa \to \infty}}\, f(\Psi(u_\kappa)) \frac{\beta}{\delta}\, \Psi'(u_\kappa).$$

But this is, by (20),

$$\leqq \frac{\beta}{\delta}\, \delta\, \overline{\lim_{\kappa \to \infty}} f(u_\kappa) \leqq \beta\, G(x).$$

20. We have therefore

$$G(\Psi(x))\,\Psi'(x) \leqq \beta G(x),$$

so that $\int^{\infty} G(x)\,dx$ is convergent. But then, since $G(x)$ is monotonically decreasing, the series $\sum^{\infty} G(\nu)$ is convergent too, and, by (24), the same holds for the series (18). The Theorem 5 is proved.

21. Theorem 6. *Assume that $\Psi(x)$ is for $x \geqq x_0$ a positive and monotonically increasing differentiable function for which we have (16). Suppose further that $f(x)$ is $> 0$ for $x \geqq x_0$, is integrable and bounded from below by a positive number in any finite subinterval of $\langle x_0, \infty)$ and satisfies for a constant $\beta > 1$ and for all $x \geqq x_0$ the condition*

$$f(\Psi(x))\,\Psi'(x) \geqq \beta f(x), \quad x \geqq x_0. \tag{25}$$

*Finally assume that there exists an* $x_1 \geqq x_0$ *such that we have for all* $x$, $u$ *with* $x \geqq u \geqq x_1$:

$$\frac{\Psi'(x)}{\Psi'(u)} \geqq \frac{1}{\beta} (x \geqq u \geqq x_1) . \tag{26}$$

*Then the series (18) is divergent.*

22. Observe that the condition (26) is certainly satisfied from a certain $x_1$ on, if $\Psi(x)$ has a finite limit $\omega$,

$$\Psi'(x) \to \omega < \infty \, (x \to \infty) . \tag{27}$$

23. *Proof of the Theorem 6.* Since $x_0$ can be replaced by any greater number we can assume, without loss of generality, that $x_1 = x_0$. Then we proceed as in the proof of the Theorem 4 defining $F(x)$ by (19) and obtain, as in the section 15, using (26):

$$F(\Psi(x))\,\Psi'(x) = \lim_{\kappa \to \infty} f(\Psi(v_\kappa))\,\Psi'(x) \geqq \frac{1}{\beta} \overline{\lim_{\kappa \to \infty}} f(\Psi(v_\kappa))\,\Psi'(v_\kappa)$$

$$\geqq \overline{\lim_{\kappa \to \infty}} f(v_\kappa) \geqq F(x) .$$

24. We see that $F(x)$ satisfies the conditions of the Theorem 2; therefore the integral $\int^{\infty} F(x)\,dx$ is divergent and the same holds for the series $\sum^{\infty} F(\nu)$, as $F(x)$ is monotonically decreasing. But then the series (18) is also divergent since $f(x)$ is a majorant of $F(x)$. The Theorem 6 is proved.

## IV. Another method in the case of divergence

25. Theorem 7. *The assertion of the Theorem 4 remains valid if the assumption that* $\Psi'(x)$ *is monotonically increasing is replaced by the assumption that* $\Psi'(x)$ *is monotonically decreasing.*

26. *Proof.* Since in any case $\Psi'(x) \geqq 0$ there exists a finite $\omega$ such that

$$\Psi'(x) \downarrow \omega \quad (x \to \infty)$$

and, as in the sec. 17, we see that this limit is $\geqq 1$.

Define the $a_\nu$ as in the sec. 18. Since we can multiply $f(x)$ by any fixed constant, we can assume that we have:

$$f(x) \geqq 1 \quad (a_0 \leqq x \leqq a_1).$$

27. Denote the inverse function of $\Psi(x)$ by $\sigma(x) \equiv \sigma_1(x)$ and its iterated $\sigma(\sigma(x))$, $\sigma(\sigma(\sigma(x)))$,... by $\sigma_2(x)$, $\sigma_3(x)$,... and define a new function $F(x)$ in such a way that we have:

$$F(x) = \frac{F(\sigma(x))}{\Psi'(\sigma(x))} \quad (x \geqq a_1). \tag{28}$$

For this purpose we put:

$$F(x) = 1 \ (a_0 \leqq x < a_1), \ F(x) = \frac{1}{\Psi'(\sigma_1(x))} (a_1 \leqq x < a_2), \ldots$$

$$F(x) = \prod_{\nu=1}^{n} \frac{1}{\Psi'(\sigma_\nu(x))} (a_n \leqq x < a_{n+1}), \tag{29}$$

and (28) follows immediately.

28. From (29) we have for $x = a_n$ and $x \uparrow a_{n+1}$:

$$F(a_n) = \prod_{\nu=0}^{n-1} \frac{1}{\Psi'(a_\nu)}, \ F(a_{n+1} - 0) = \prod_{\nu=1}^{n} \frac{1}{\Psi'(a_\nu)},$$

and, putting $\frac{1}{\Psi'(a_0)} = \sigma_0 \leqq 1$:

$$\frac{F(a_n)}{F(a_n - 0)} = \sigma_0 = \frac{1}{\Psi'(a_0)}. \tag{30}$$

Since we have $\omega \geqq 1$, $\Psi'(x) \geqq 1$, we see that $\Psi(x) - x$ is non-decreasing, and therefore, the same holds for the length of the $n$ – th interval between the $a_\nu$, $a_{n+1} - a_n$. The number of the $a_\nu$ lying in an interval of the length 1 in the half-line $x \geqq a_0$ has a finite upper bound which may be denoted by $k$.

29. From (29) it follows obviously that $F(x)$ is continuous and monotonically increasing in any half-open interval $\langle a_n, a_{n+1})$. In the points $a_\nu$ we have a discontinuity if $\sigma_0 \neq 1$. We can therefore write for any $y \geqq a_0$:

$$F(y) \geqq \sigma_0^k F(x) \quad (y - 1 \leqq x \leqq y). \tag{31}$$

30. Take here as $y$ an integer $m \geqq a_0$, multiply by $dx$ and integrate from $m - 1$ to $m$; we obtain

$$F(m) \geqq \sigma_0^k \int_{m-1}^{m} F(x)\, dx$$

and therefore, denoting by $n_0 - 1$ the first integer $> a_0$:

$$\sum_{v=n_0}^{n} F(v) \geqq \sigma_0^k \int_{n_0-1}^{n} F(x)\, dx\,.$$

31. On the other hand, the relation (28) can be written as

$$F\left(\Psi(x)\right)\Psi'(x) = F(x)\,, \tag{32}$$

and it follows therefore from the Theorem 2 that $\int^{\infty} F(x)\, dx$ is divergent. We see that the series $\sum^{\infty} F(\nu)$ diverges too.

32. In order to prove our Theorem it is therefore sufficient to prove that we have

$$f(x) \geqq F(x) \quad (x \geqq a_0)\,. \tag{33}$$

But this relation is evident in the interval $\langle a_0, a_1)$. Comparing (17) and (32) this inequality follows also for the interval $\langle a_1, a_2)$ and from there on by induction for any $x \geqq a_0$. The Theorem 7 is proved.

## V. New conditions for the Euler-Maclaurin Theorem

33. One of the ideas underlying the proof of the Theorem 6 was the introduction of the condition (26) which is a kind of weakened monotony condition.

We give in what follows the corresponding generalisation of the Euler-Maclaurin convergence criterion, in which we try to weaken the monotony condition even more. Combining the conditions of the Theorem 8 with the assumptions of the Theorems 1 and 2 we obtain then further criteria for the convergence and divergence of the series (18).

34. THEOREM 8. *Let $\gamma$, $\varepsilon$, $K$ be fixed positive numbers, and $A$ a fixed real number. Assume $f(x)$ non-negative and integrable in any finite subinterval of the interval $\langle x_0, \infty)$. If for any $y \geqq Max\left(x_0, \frac{x_0 - A}{\gamma}\right)$ we have*

$$f(y) \leqq Kf(x) \quad (\gamma y + A \leqq x \leqq \gamma y + A + \varepsilon), \tag{34}$$

*then from the convergence of the integral (2) follows the convergence of the series (18).*

*If for any $x \geqq \text{Max}\left(x_0, \frac{x_0 + \gamma - A}{\gamma}\right)$ we have*

$$f(x) \geqq Kf(y) \quad (\gamma x + A - \gamma \leqq y \leqq \gamma x + A), \tag{35}$$

*then from the divergence of the integral (2) follows the divergence of the series (18).*

35. *Proof.* If (34) holds we have, taking as $y$ an integer $v$ and integrating with respect to $x$ from $\gamma v + A$ to $\gamma v + A + \varepsilon$:

$$f(v) \leqq \frac{K}{\varepsilon} \int_{\gamma v + A}^{\gamma v + A + \varepsilon} f(x)\, dx,$$

and therefore, denoting by $n_0$ a convenient integer, for any $n > n_0$:

$$\frac{\varepsilon}{K} \sum_{v=n_0}^{n} f(v) \leqq \sum_{v=n_0}^{n} \int_{\gamma v + A}^{\gamma v + A + \varepsilon} f(x)\, dx. \tag{36}$$

36. The limits of the integration in the right hand integrals lie here between $\gamma n_0 + A$ and $\gamma n + A + \varepsilon$.

If an $x$ lies in one of the integration intervals in (36) we have

$$\gamma v + A \leqq x \leqq \gamma v + A + \varepsilon, \frac{x - A}{\gamma} - \frac{\varepsilon}{\gamma} \leqq v \leqq \frac{x - A}{\gamma}$$

and we see that any such $x$ can lie at the most in $\frac{\varepsilon}{\gamma} + 1$ such intervals. The right hand expression in (36) is therefore

$$\leqq \left(\frac{\varepsilon}{\gamma} + 1\right) \int_{\gamma n_0 + A}^{\gamma n + A + \varepsilon} f(x)\, dx$$

and our assertion corresponding to the condition (34) is proved.

37. Assuming that (35) is satisfied we take $x$ as an integer $v$ and obtain, integrating with respect to $y$ from $\gamma v + A - \gamma$ to $\gamma v + A$:

$$\gamma f(x) \geqq K \int_{\gamma v + A - \gamma}^{\gamma v + A} f(y)\, dy ,$$

and therefore, for a convenient integer $n_0$,

$$\frac{\gamma}{K} \sum_{v=n_0}^{n} f(v) \geqq \sum_{v=n_0}^{n} \int_{\gamma v + A - \gamma}^{\gamma v + A} f(y)\, dy = \int_{\gamma n_0 + A - \gamma}^{\gamma n + A} f(y)\, dy .$$

From this inequality the assertion corresponding to the condition (35) follows immediately. The Theorem 8 is proved.

38. Corollary. *Assume $f(x)$ non-negative, finite and integrable in any finite subinterval of $\langle x_0, \infty)$. If there exists an integer N such that $x^N f(x)$ is from a certain $x$ on either monotonically increasing or monotonically decreasing, the series (18) converges or diverges according as the integral (2) is convergent or divergent.*

## VI. Comments on Pringsheim's discussion of the problem

39. Although Ermakof's convergence and divergence criteria and in particular Ermakof's second proof, using Abel's functional equation, are extremely interesting, they remained very little known and it appears that the author's paper [5] was the first in which the problem was taken up in a modern way. The reason for this may lie partly in the very negligent way in which Ermakof's notes were written and partly in some erroneous and misleading statements about this problem which were formulated by Pringsheim in [6], [7] and [8]. Although the essential merit of Ermakof's second paper consists just in the fact that the function $f(x)$ need not be assumed as monotonic — it is true that Ermakof does not even mention this point in [2] — Pringsheim says in [7], pp. 308-309: -"Es ist mir neuerdings gelungen, dieselben [*that is Ermakof's criteria*] von einer ihnen (auch in der von Herrn Ermakoff gegebenen Darstellung)

anhaftenden, sehr wesentlichen Beschränkung, nämlich der *ausschliesslichen* Anwendbarkeit auf Reihen mit *niemals zunehmenden* Gliedern zu befreien, und zwar lassen sie sich auch in dieser *erweiterten* Form mit Hilfe der oben charakterisierten, in meiner Abhandlung durchgeführten Methode ableiten."

Further, on p. 327 of [7] Pringsheim says after having discussed the case that $f(x)$ is never increasing, in a footnote: " Dies ist der von Herrn Ermakoff ausschliesslich betrachtete Fall."

The same is implied in the statement about Ermakof's criteria in [8] on p. 89: " Die letztere habe ich neuerdings in der Weise verallgemeinert, dass $f(x)$ nicht mehr als *monoton* vorausgesetzt zu werden braucht."

It is obvious that the reader of the last statement cannot help believing that while Ermakof did assume the monotony of $f(x)$, Pringsheim in his paper [7] quoted proved that this assumption can be dropped.

On the other hand, what Pringsheim did in [7] with Ermakof's criteria can be reduced to the observation that the transition from (2) to (18) in the Euler-Maclaurin theorem can be achieved if we have

$$\frac{f(\nu+\theta)}{f(\nu)} \to 1 \; (\nu \to \infty \,, \quad 0 \leqq \theta \leqq 1)$$

for natural $\nu$, uniformly in $\theta$.

This is certainly a pretty unfair way to deal with the ingenious proof of Ermakof and the beautiful result given in his paper [2].

40. However, Pringsheim derived the above result which is, of course, a very special case of our Theorem 8, from an elegant " Corollary " to Ermakof's criteria. In this " Corollary " the expression (1) is replaced by:

$$\frac{f([\Psi(x)])\,\Psi'(x)}{f([x])},$$

where, as usual, $[x]$ denotes the greatest integer contained in $x$.

This expression is of interest since only the values of $f$ for integer arguments enter into it, and in this discussion the assumption about the monotony of $f(x)$ is not necessary. On the other hand, it is pretty difficult to handle if $f(x)$ is given by an analytic expression.

Since Pringsheim's formulation of this " Corollary " is too special we derive in the sections 44-45 a generalized form of it.

41. In the paper [6] Pringsheim gives a very general convergence criterion which is also mentioned in [7] and [8]. This criterion uses, in the notations of sec. 7-12 and assuming that $\psi(x) \leqq \Psi(x)$ $(x \geqq x_1)$, the expression

$$\varphi_h(x) = \int_{\Psi(x)}^{\Psi(x+h)} f(x)\,dx \Bigg/ \int_{\psi(x)}^{\psi(x+h)} f(x)\,dx \qquad (37)$$

for a fixed $h > 0$. Pringsheim proves, that if $\lim\limits_{x\to\infty} \varphi_h(x)$ is $> 1$, the integral (2) diverges, while this integral is convergent if $\lim\limits_{x\to\infty} \varphi_h(x) < 1$.

In quoting this result in [7] and [8] Pringsheim says that Ermakof's result follows from his for $h \to 0$. This is, of course, not correct since in this passage to the limit something like the uniform differentiability of $\Psi(x)$ in the infinite interval $(x, \infty)$ has to be used. As a matter of fact, Pringsheim mentiones this restriction in his first publication [6], while in [7] and [8] any reference to this restriction is omitted.

42. We give in what follows a proof of Pringsheim's criterion in a generalized form, avoiding the assumption that $\lim\limits_{x\to\infty} \varphi_h(x)$ exists. We prove:

*If for a positive* $\varepsilon$ *from an* $x = x_1$ *on we have*

$$\varphi_h(x) \geqq 1 + \varepsilon \quad (x \geqq x_1), \qquad (38)$$

*the integral (2) diverges, while this integral converges if we have from an* $x \geqq x_1$ *on:*

$$\varphi_h(x) \leqq 1 - \varepsilon \quad (x \geqq x_1). \qquad (39)$$

*Proof*. From (38) follows obviously for a natural $n$:

$$1+\varepsilon \leqq \sum_{v=1}^{n} \int_{\Psi(x_1+(v-1)h)}^{\Psi(x_1+vh)} f(x)\,dx \Bigg/ \sum_{v=1}^{n} \int_{\psi(x_1+(v-1)h)}^{\psi(x_1+vh)} f(x)\,dx$$

$$= \int_{\Psi(x_1)}^{\Psi(x_1+nh)} f(x)\,dx \Bigg/ \int_{\psi(x_1)}^{\psi(x_1+nh)} f(x)\,dx.$$

If we replace in the numerator of the right hand quotient $\Psi(x_1)$ by $\psi(x_1) \leqq \Psi(x_1)$ this quotient is not decreased and we have

$$\int_{\psi(x_1)}^{\Psi(x_1+nh)} f(x)\,dx \Bigg/ \int_{\psi(x_1)}^{\psi(x_1+nh)} f(x)\,dx \geqq 1+\varepsilon.$$

Therefore the integral (2) is divergent, because otherwise the left hand quotient would tend to 1 with $n \to \infty$.

43. Under the condition (39) we have again for a natural $n$:

$$1-\varepsilon \geqq \int_{\Psi(x_1)}^{\Psi(x_1+nh)} f(x)\,dx \Bigg/ \int_{\psi(x_1)}^{\psi(x_1+nh)} f(x)\,dx$$

$$\geqq \int_{\Psi(x_1)}^{\Psi(x_1+nh)} f(x)\,dx \Bigg/ \int_{\psi(x_1)}^{\Psi(x_1+nh)} f(x)\,dx.$$

Therefore the integral (2) must converge, since otherwise the right hand quotient would tend to 1 with $n \to \infty$. Combining this result with the Theorem 8 we obtain again criteria for the convergence and divergence of (18).

44. Pringsheim derived in his paper [7] the "Corollary" from Ermakof's results, quoted above, in the following way.

If the function $f(x)$ is defined for all integers $\nu \geqq \nu_0$, define the function $\varphi(x)$ by

$$\varphi(x) \equiv f([x]) \quad (x \geqq \nu_0). \tag{40}$$

Then we have for an integer $n$ which is $\geqq$ than an integer $\nu_0$,

$$\int_{\nu_0}^{n+1} \varphi(x)\,dx = \sum_{\nu=\nu_0}^{n} f(\nu), \tag{41}$$

and conditions for the convergence or divergence of the series (18) are obtained, applying to the integral $\int^{\infty} \varphi(x)\,dx$ Ermakof's criteria. In this way we obtain corresponding criteria without assuming anything about the monotony of $f(x)$.

45. As a matter of fact Pringsheim formulates only the condition

$$\lim_{x\to\infty} \frac{f([\Psi(x)])\,\Psi'(x)}{f([\psi(x)])\,\psi'(x)} < 1$$

for the convergence and

$$\lim_{x\to\infty} \frac{f([\Psi(x)])\,\Psi'(x)}{f([\psi(x)])\,\psi'(x)} > 1$$

for the divergence, where $\Psi(x)$ and $\psi(x)$ are assumed to tend monotonically to $\infty$ with $x \to \infty$ and to satisfy $\Psi(x) > \psi(x)$. However, it is obvious, e.g. from the corresponding specialisations of our Theorems 1 and 2 that we can use

$$f([\Psi(x)])\,\Psi'(x) \leqq \alpha f([\psi(x)])\,\psi'(x), \quad \alpha < 1 \tag{42}$$

as convergence condition and

$$f([\Psi(x)])\,\Psi'(x) \geqq f([\psi(x)])\,\psi'(x) \tag{43}$$

as that for divergence.

Incidently, it is clear that we have in these cases the same degree of generality if we take $\psi(x) \equiv x$.

46. Applying the same idea directly to the Theorems 1—3 we have the following three Theorems in which we assume that $\psi(x)$ and $\Psi(x)$ are totally continuous for $x \geqq \nu_0$ and that $f(\nu)$ is defined and $\geqq 0$ for all integers $\nu \geqq \nu_0$.

THEOREM 1. *Assume that we have (4) for a sequence* $b_\nu \geqq \nu_0$ $(\nu = 1,2,...)$. *Then, if we have (42) for almost all* $x \geqq \nu_0$ *and for a positive* $\alpha < 1$ *the series (18) is convergent.*

Further, assuming that $f(\nu)$ is not $= 0$ for all sufficiently great integers $\nu$, we have for all $x \geqq \nu_0$:

THEOREM 2. *Assume that there exists an* $a \geqq \nu_0$ *and an integer* $\nu_1 \geqq \nu_0$ *such that:*

$$\Psi(a) > \nu_1 \geqq \psi(a)\,,\; f(\nu_1) > 0\,, \tag{44}$$

*and a sequence* $b_\nu \geqq \nu_0$ $(\nu = 1,2,...)$ *such that we have (8). Then, if (43) holds for almost all* $x \geqq \nu_0$, *the series (18) is divergent and we have (10) for all* $x \geqq a$.

THEOREM 3. *Assume that there exists a constant* $\gamma$, $0 < \gamma < 1$, *and a sequence* $b_\nu \geqq \nu_0$ *such that (13) holds and further that for a constant* $c$ *and for all integers* $\nu \geqq \nu_1$ *we have:*

$$\nu f(\nu) \leqq c \quad (\nu \geqq \nu_1)\,.$$

*If then (42) holds for a certain* $\alpha < 1$ *the series (18) is convergent, and the relation* $\Psi(a) \leqq \psi(a)$ *is for an* $a \geqq \nu_0$ *only possible, if* $f(\nu) = 0$ *for all* $\nu \geqq [\Psi(a)]$.

Observe that in applying the Theorems 1′, 2′ and 3′ to $\varphi$ the transformation formula can be certainly applied since $|\varphi(x)|$ is uniformly bounded.

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