EUCLIDEAN ALGORITHMS AND MUSICAL THEORY

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Objekttyp: **Article**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **10 (1964)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

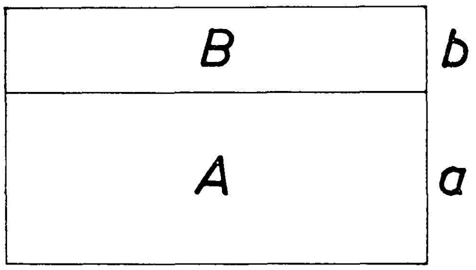
PDF erstellt am: **25.04.2024**

Persistenter Link: https://doi.org/10.5169/seals-39413

EUCLIDEAN ALGORITHMS AND MUSICAL THEORY [1])

by Viggo BRUN

Aristotle has an interesting remark in his Topica. He says that the ratio between the areas A and B is the same as between a and b.



$$A:B = a:b$$

He also says that it is not easy to prove this without having a definition of equality between two ratios (they could be irrational). And he gives the definition: they are equal because they have the same *Antanairesis*. The meaning of this word is " to take away in turn ". Van der Waerden contends that *Theaitetos* has given this definition of the equality of two ratios and that *Euclid* has replaced it by the famous definition of *Eudoxos* (later used by Dedekind). It is remarkable that Omar Khayyam, the Persian poet and mathematician, uses the same definition as Aristotle.

However, Euclid also treats the method of " taking away in turn " which has later been called the Euclidean Algorithm. Today this algorithm is usually given as an algorithm of divisions (alternatively represented by a continued fraction). But

[1]) Ten minutes lecture given at the International Congress of Mathematicians in Stockholm 1962.

Euclid himself treated his algorithm as an algorithm of subtraction — that is, as an Antanairesis. He nevertheless did not introduce this word in his elements. I shall give this algorithm of Euclid, "the taking away in turn", applied to two given real positive numbers a and b (where $a > b$) the following form:

$$
\begin{array}{r|cc}
a & 1 & 0 \\
b & 0 & 1 \\
\hline
a-b & 1 & 0 \\
b & 1+0 & 0+1 \\
\hline
--- & --- & \\
--- & --- & \\
\hline
a_r & x_r & y_r \\
b_r & x_r' & y_r' \\
\hline
a_r - b_r & x_r & y_r \\
b_r & x_r + x_r' & y_r + y_r' \\
\hline
\multicolumn{3}{c}{-----------}
\end{array}
$$

On the left-hand side we have taken the difference between the greatest and the smallest number and we have repeated the smallest one. Of the two new numbers $a-b$ and b, one must be greater than (or equal to) the other one and the process therefore can continue. On the right hand side we have repeated the first line and we have added the numbers in the first and in the second line to get the numbers in the new second line.

On the right hand side we get fractions $\frac{y_r}{x_r}$ to approximating $\frac{a}{b}$, as is well known from the theory of continued fractions.

In 1919 I generalized this algorithm for two numbers to an algorithm for three numbers a, b and c (where $a > b > c$). This algorithm has the following form:

a	1	0	0
b	0	1	0
c	0	0	1
$a-b$	1	0	0
b	$1+0$	$0+1$	$0+0$
c	0	0	1
— — —	— —	— —	— —
— — —	— —	— —	— —
a_r	x_r	y_r	z_r
b_r	x_r'	y_r'	z_r'
c_r	x_r''	y_r''	z_r''
a_r-b_r	x_r	y_r	z_r
b_r	x_r+x_r'	y_r+y_r'	z_r+z_r'
c_r	x_r''	y_r''	z_r''
— — —	— —	— —	— —
— — —	— —	— —	— —

On the left-hand side we have taken the difference between the greatest and the next-greatest number and we have repeated the next-greatest and the smallest numbers. On the right-hand side we have repeated the first and the third line and we have replaced the second line by the sum of the numbers in the first two lines.

My algorithm makes it possible to treat the following problem: Three real, positive numbers a, b and c are given. Find three integers x_r, y_r and z_r which satisfy the approximative relations

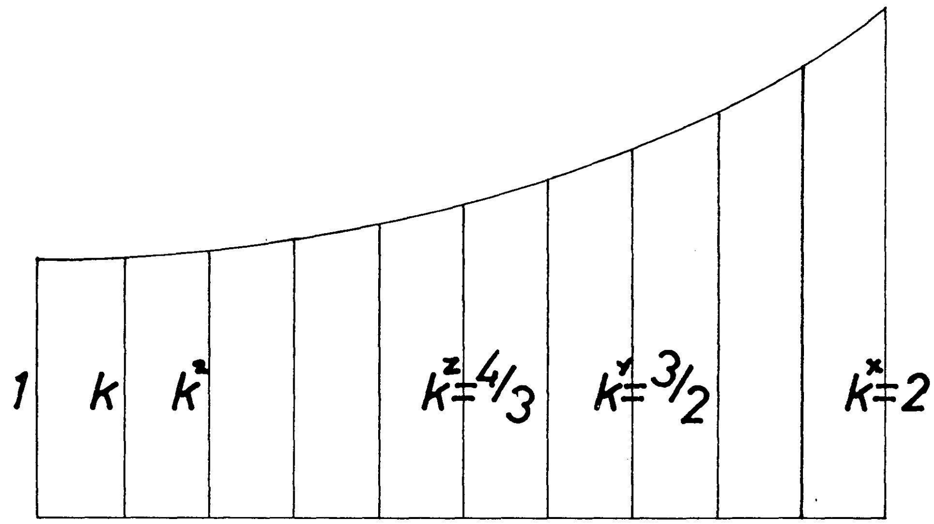
$$\frac{a}{x_r} \approx \frac{b}{y_r} \approx \frac{c}{z_r} \cdot$$

I have proved various theorems concerning this algorithm, for instance that the procedure is convergent:

$$\frac{y_r}{x_r} \to \frac{b}{a}, \qquad \frac{z_r}{x_r} \to \frac{c}{a} \cdot$$

My algorithm seems to be able to give a certain contribution to musical theory.

From ancient times it has been a problem to construct a stringed instrument where the last string had the double length of the first one, and with the property that the ratio between one string and the preceding one was to be constant.



It was also desired that one string should be nearly $\frac{3}{2}$ times and one nearly $\frac{4}{3}$ times as long as the first string. We then have to find three integers x, y and z such that

$$k^x \approx 2\,, \qquad k^y \approx \frac{3}{2}\,, \qquad k^z \approx \frac{4}{3}\,,$$

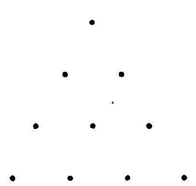
or

$$\frac{\log 2}{x} \approx \frac{\log \frac{3}{2}}{y} \approx \frac{\log \frac{4}{3}}{z}\,.$$

The fact that the first four integers 1, 2, 3 and 4 were sufficient to write the four fractions

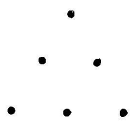
$$\frac{1}{1}\,, \frac{4}{3}\,, \frac{3}{2}\,, \frac{2}{1}\,,$$

was symbolized by the ancient Greeks by the diagram " tetraktys ":



This diagram was only a symbol for the four numbers 1, 2, 3 and 4,—but to the Greeks it was a sacred symbol. It gives us a good idea how to systematize our research. First we will study the logarithms of all fractions (> 1 and ≦ 2) with numerators and denominators among the numbers 1, 2, and 3, then among 1, 2, 3 and 4, and so on. In each case I add a generalized tetraktys to remind of the Greek influence on the treatment of our problem.

Problem 1. Find integers x and y such that

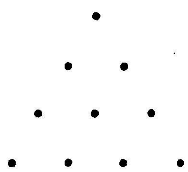


$$\frac{\log 2}{x} \approx \frac{\log \frac{3}{2}}{y} .$$

This problem has been treated by *L. Euler* who used continued fractions to solve it. He found for instance

$$x = 12 , \qquad y = 7 \qquad \textit{and} \qquad x = 17 , \qquad y = 10 .$$

Problem 2: Find three integers such that



$$\frac{\log 2}{x} \approx \frac{\log \frac{3}{2}}{y} \approx \frac{\log \frac{4}{3}}{z} .$$

Here it is natural to use my algorithm, but as

$$\log 2 - \log \frac{3}{2} = \log \frac{4}{3} ,$$

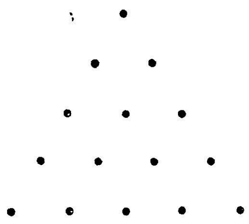
the algorithm at once reduces to the algorithm employed under problem 1. The solutions for x and y will be the same, and we get for instance

$$x = 12 , \qquad y = 7 , \qquad z = 5 ,$$

and

$$x = 53 , \qquad y = 31 , \qquad z = 22 .$$

Problem 3: To find integers x, y, z, u, v such that



$$\frac{\log 2}{x} \approx \frac{\log \frac{5}{3}}{y} \approx \frac{\log \frac{3}{2}}{z} \approx \frac{\log \frac{4}{3}}{u} \approx \frac{\log \frac{5}{4}}{v}.$$

As, however

$$\log 2 - \log \frac{4}{3} = \log \frac{3}{2},$$

and

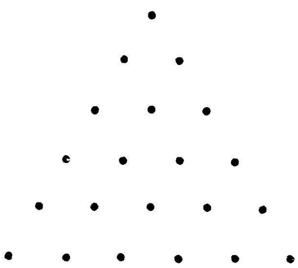
$$\log \frac{5}{3} - \log \frac{5}{4} = \log \frac{4}{3},$$

my algorithm for five numbers will soon reduce to an algorithm for three numbers. I mention two of the solutions:

$$x = 31, \quad y = 23, \quad z = 18, \quad u = 13, \quad v = 10;$$

$$x = 53, \quad y = 39, \quad z = 31, \quad u = 22, \quad v = 17.$$

Problem 4: To find integers x, y, z, u, v, w such that



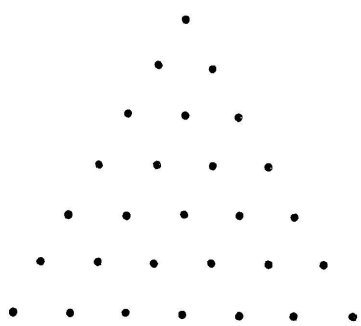
$$\frac{\log 2}{x} \approx \frac{\log \frac{5}{3}}{y} \approx \frac{\log \frac{3}{2}}{z} \approx \frac{\log \frac{4}{3}}{u} \approx \frac{\log \frac{5}{4}}{v} \approx \frac{\log \frac{6}{5}}{w}.$$

As, however

$$\log 2 - \log \frac{5}{3} = \log \frac{6}{5},$$

the algorithm will after a while reduce to the former one, and will give the same solutions.

Problem 5: To find integers x, y, z, u, v, w, p, q, r such that



$$\frac{\log 2}{x} \approx \frac{\log \frac{7}{4}}{y} \approx \frac{\log \frac{5}{3}}{z} \approx \frac{\log \frac{3}{2}}{u} \approx \frac{\log \frac{7}{5}}{v} \approx$$

$$\frac{\log \frac{4}{3}}{w} \approx \frac{\log \frac{5}{4}}{p} \approx \frac{\log \frac{6}{5}}{q} \approx \frac{\log \frac{7}{6}}{r}.$$

My algorithm for these nine numbers will for example give the solutions

$x = 12,\ x = 31,\ x = 72,\ x = 87,\ x = 91,\ x = 99,\ x = 171,\ \ldots,$

all of which give different possibilities for partitioning the octave.

My algorithm for these nine numbers will after a while reduce to an algorithm for four numbers. The reason is that we have four primes (2, 3, 5, 7) not surpassing the number 7. Historically it is of interest to remark that the division of the octave into 31 parts has been proposed by *Vincentino*, *Mersenne*, *Huygens* and presently by *Fokker*. The number 53 is said to have been known by *Philolaos*, a pupil of *Pythagoras*, and later on to have been used in China. *Mercator* has especially suggested the number 53. The number 72 is not mentioned in Barbour's excellent book " Tuning and Temperament, A Historical Survey ". But the number 74 suggested by *Drobisch* is mentioned. The number 72 is nevertheless much better than 74. I have read in a journal that a Russian scientist, *J. Mursin*, for many years has done experiments with a musical instrument where the octave is divided into 72 parts.

Additional remarks

As I hope that the reader has to his disposal more than the ten minutes that I had for my lecture in Stockholm, it will perhaps be appropriate to give some additional remarks.

1. In " Science Awakening ", van der Waerden has discussed the remark on Antanairesis of Aristotle in his Topica.

2. Omar Khayyam (Alhajjâmî), 1044-1123. His treatment of proportions can be found in *D. S. Kasir:* The Algebra of Omar Khayyam (New York, 1931) and in *A. P. Juschkewitsch* und *B. A. Rosenfeld*: Die Mathematik der Länder des Ostens in Mittelalter (p. 134), Veb., Berlin.

3. If it is desired to give the Euclidean subtraction algorithm—the Antanairesis— the form of a continued fraction, it can be done as in the following example:

$$\frac{10}{7} = 1 + \cfrac{1}{2 + \cfrac{1}{3}} = 1 + \cfrac{1}{1 + 1 + \cfrac{1}{1 + 1 + 1}} \cdot$$

If we here break off the fractions before a plus sign, we get in both cases good approximation to $\frac{7}{10}$.
In the first case we get

$$\frac{1}{1}, \frac{3}{2}, \frac{10}{7}.$$

In the second case we get

$$\frac{1}{1}, \frac{2}{1}, \frac{3}{2}, \frac{4}{3}, \frac{7}{5}, \frac{10}{7}.$$

We can express it in this manner: In the first case many good approximations are " jumped over ".

4. My first works on the generalization of continued fractions were based on a geometrical interpretation. My algorithm is different from that of *Poincaré*, who also used geometrical considerations. In my generalization I have used only subtractions, and not divisions as did Jacobi.

5. My algorithm and analogous ones have afterwards been treated by *Nils Pipping*, *Marcel David* and *Ernst Selmer*. Pipping has studied the ramified algorithm

$$\begin{array}{cc} \multicolumn{2}{c}{a} \\ \multicolumn{2}{c}{b} \\ \multicolumn{2}{c}{c} \\ \hline a-b & a-c \\ b & b \\ c & c \end{array}$$

and Selmer has studied the algorithm

$$\begin{array}{c} a \\ b \\ c \\ \hline a-c \\ b \\ c \end{array}$$

(for $c > 0$).

6. I have also studied the analogous algorithm for four numbers, which has many—but not all—of the properties of the algorithm for three numbers. The analogous algorithm for five or more numbers has not been appropiately studied at all. Here it seems to be much to do for younger mathematicians.

7. J. M. Barbour was the first one to give a systematic solution of the problem to find integers x, y, z such that

$$\frac{\log 2}{x} \approx \frac{\log \frac{3}{2}}{y} \approx \frac{\log \frac{5}{4}}{z}.$$

He used the method of Jacobi, but had to make two modifications.

8. The most primitive method to solve the problem

$$\frac{\log 2}{x} \approx \frac{\log \frac{3}{2}}{y}$$

in integers x and y would be to choose successively $x = 1$, $x = 2$, $x = 3$ etc. and afterwards choose y as the nearest integer to

$$y' = \frac{\log \frac{3}{2}}{\log 2} x .$$

The theory of continued fractions can here shorten the process considerably. This theory could also—if desired— be modified in such a way that only the numbers x had to be calculated directly by the mentioned algorithm, and afterwards the numbers y choosen as the nearest integers to y'. The degree of accuracy will then be seen directly. We obtain for example

$$\frac{\log 2}{12} = \frac{\log \frac{3}{2}}{7{,}02\ldots},$$

and

$$\frac{\log 2}{53} = \frac{\log \frac{3}{2}}{31{,}003\ldots}.$$

We then have to choose $y = 7$ and $y = 31$.

9. This method can be used also for my generalized algorithms. As an example I mention problem 2, to find three integers x, y, z such that

$$\frac{\log 2}{x} \approx \frac{\log \frac{3}{2}}{y} \approx \frac{\log \frac{4}{3}}{z}\,.$$

My algorithm for these three numbers can be given in the following form, when in the first place we only want to find the numbers x (and afterwards determine y and z):

	x								
$\log 2 = 0{,}301$	1								
$\log \frac{3}{2} = 0{,}176$	0								
$\log \frac{4}{3} = 0{,}125$	0								
0,125	1	0,176	1						
0,176	1	0,125	1						
0,125	0	0,125	0						
		0,051	1	0,125	2				
		0,125	2	0,125	0				
		0,125	0	0,051	1				
				0,000	2	0,125	2		
				0,125	2	0,051	1		
				0,051	1	0,000	2		
						0,074	2		
						0,051	3		
						0,023	2	0,051	5
						0,051	5	0,023	2
								0,028	5
								0,023	7
								0,005	5
								0,023	12

The continuation of the calculation will give

$$x = 17\,,\ x = 29\,,\ x = 41\,,\ x = 53 \quad \text{etc.},$$

and we obtain for example

$$\frac{\log 2}{12} = \frac{\log \frac{3}{2}}{7{,}02\,..} = \frac{\log \frac{4}{3}}{4{,}98\,..},$$

and

$$\frac{\log 2}{53} = \frac{\log \frac{3}{2}}{31{,}003\,..} = \frac{\log \frac{4}{3}}{21{,}997\,..}.$$

We can therefore choose for instance

$$x = 12, \quad y = 7, \quad z = 5,$$

or

$$x = 53, \quad y = 31, \quad z = 22.$$

10. As my algorithm is very little studied for more than four numbers, the alternative method consisting of first determining only x by the algorithm, and afterwards $y, z \dots ..$, will be preferable in these cases. For problem 5 we find for example 72 as a value for x, and we then get

$$\frac{\log 2}{72} = \frac{\log \frac{7}{4}}{58{,}13\,..} = \frac{\log \frac{5}{3}}{53{,}06\,..} = \frac{\log \frac{3}{2}}{42{,}12\,..} = \frac{\log \frac{7}{5}}{34{,}95\,..} =$$

$$\frac{\log \frac{4}{3}}{29{,}88\,..} = \frac{\log \frac{5}{4}}{23{,}18\,..} = \frac{\log \frac{6}{5}}{18{,}94\,..} = \frac{\log \frac{7}{6}}{16{,}01\,..}.$$

It may be of interest to compare this result with the corresponding result for the number $x = 74$ which Drobisch proposed for dividing the octave:

$$\frac{\log 2}{74} = \frac{\log \frac{7}{4}}{59{,}75\,..} = \frac{\log \frac{5}{3}}{54{,}54\,..} = \frac{\log \frac{3}{2}}{43{,}29\,..} = \frac{\log \frac{7}{5}}{35{,}92\,..} =$$

$$\frac{\log \frac{4}{3}}{30{,}71\,..} = \frac{\log \frac{5}{4}}{23{,}82\,..} = \frac{\log \frac{6}{5}}{19{,}46\,..} = \frac{\log \frac{7}{6}}{16{,}46\,..}.$$

From this it is seen that the number 72 is much better than 74 for dividing the octave.

From the relation

$$\frac{\log 2}{99} = \frac{\log \frac{7}{4}}{79{,}93\,..} = \frac{\log \frac{5}{3}}{72{,}96\,..} = \frac{\log \frac{3}{2}}{57{,}91\,..} = \frac{\log \frac{7}{5}}{48{,}06\,..} =$$

$$\frac{\log \frac{4}{3}}{41{,}09\,..} = \frac{\log \frac{5}{4}}{31{,}87\,..} = \frac{\log \frac{6}{5}}{26{,}04\,..} = \frac{\log \frac{7}{6}}{22{,}02\,..}\,.$$

it will be seen that a division of the octave into 99 parts is also of great advantage. Barbour has not mentioned this possibility but he has mentioned 98, which however gives less satisfactory results.

11. At the congress in Stockholm, *A. A. Granadoss* gave a lecture, " Fechner quantum and equal temperament,", where he maintained that, considering the limited sensitivity of the ear, it will generally be unnecessary to divide the octave into more than 120 equal parts.

12. After my lecture in Stockholm, van der Waerden expressed his great scepsis concerning the assertion that Philolaos should have known the division of the octave into 53 parts. My source was Barbour, " Tuning and Temperament ", p. 123: " The most important system after the 31—is the 53 division. In theory it is also the most ancient. According to *Boethius* Pythagoras' disciple Philolaos held that ..... the tone is divisible into four diaschismata plus a comma. If, however, the diaschisma is taken as two commas exactly, the tone is divided into nine commas ". In a letter to me (Sept. 12, 1962) van der Waerden writes: " I have checked the Philolaos quotation and found that it does not say that a diaschisma is two commas. Indeed the statement diaschisma = 2 commas would be contradictory to Pythagorean musical theory. So we cannot attribute it to Philolaos."

LITERATURE

BRUN, Viggo: En generalisation av kjedebrøken I—II, *Norske Vid. Selsk. Skr.*, 1919-20, Oslo (avec des résumés en français).

—— *Algorithmes euclidiens pour trois et quatre nombres.* XIII Congr. Math. Scand. tenu à Helsinki, août 1957.

BRUN, Viggo: *Mehrdimensionale Algorithmen, welche dieEu'lersche Kettenbruchentwicklung der Zahl e verallgemeinern.* Leonhard Euler zum 250. Geburtstag, 1959, Akademie-Verlag, Berlin.

—— Music and ternary continued fractions, *Det kgl. norske vidensk. selskab. Forh.*, 23 (1950).

—— Musikk og Euklidske algoritmer, *Nordisk Matem. Tidskrift*, B.9, Oslo 1961.

PIPPING, Nils: Approximation zweier reellen Zahlen durch rationale Zahlen mit gemeinsamen Nennern. *Acta Acad.*, Åbo 1957.

—— Un critérium pour les nombres algébriques réels, fondé sur une généralisation directe de l'algorithme d'Euclide. *C.R. Acad. Sci. Paris*, 170 (1920), p. 1155.

MARCEL, David: Contribution à l'étude algorithmique des approximations rationelles simultanées de deux irrationelles, application au cas cubique. *Publ. Sci. Univ. Alger.*, sér. A, Tome III, 1956.

JACOBI, C. G. J.: Allgemeine Theorie der Kettenbruchähnlichen Algorithmen, in welchen jede Zahl aus drei vorhergehenden gebildet wird. *J. reine angew. Math.*, 69, 29-64; Werke VI 385-426.

POINCARÉ, H.: Sur une généralisation des fractions continuées. *C.R. Acad. Sci. Paris*, 99 (1884).

BARBOUR, J. M.: *Tuning and Temperament.* Michigan State College Press, 1953. Reprinted 1961.

—— Music and ternary continued fractions. *Amer. Math. Monthly 45* (1948), 545.

SELMER, E.: Om flerdimensjonal kjedebrøk. Nordisk Matem. *Tidskrift*, B.9, Oslo, 1961.

MANDELBAUM, M. J.: *Multiple division of the Octave and the tonal resources of 19-tone temperament.* Indiana University Ph.D., 1961. Music. (Univ. Microfilms, Inc. Ann Arbor, Michigan).

FOKKER, A. D.: Les mathématiques et la musique, trois conférences. *Archives du Musée Teyler.* La Haye, Martinus Nijhoff, 1947.

—— *Multiple Antanairesis.* Koninkl. Nederl. Akademie van Wetenschappen, Amsterdam, Proc. Ser. A, 66, communicated October 27, 1962.

BOETIUS: *Ueber die Musik*, übers. von O. Paul. Leipzig 1872.

VAN DER WAERDEN: *Science Awakening.*

—— Die Harmonielehre der Pythagoreer. *Hermes*, 78, pp. 163-199.